ANDERSON

Principles & Applications
Of Single Limits

Mathematics
A. M.
1904

UNIVERSITY
OF
ILL. LIBRARY

UNIVERSITY OF ILLINOIS
LIBRARY

BOOK 1904 CLASS An24 VOLUME

767
65

THE PRINCIPLES AND APPLICATIONS OF SINGLE LIMITS

BY

MARY ANDERSON, A. B., '03

THESIS FOR THE DEGREE OF MASTER OF ARTS
IN THE GRADUATE SCHOOL
UNIVERSITY OF ILLINOIS

PRESENTED, JUNE, 1904

1894 E. Elliott

UNIVERSITY OF ILLINOIS

May 26 1904

THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Mary Anderson

ENTITLED The Principles and Applications of

Single Limits

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

OF Master of Arts.

L. W. Shattuck

HEAD OF DEPARTMENT OF Mathematics.

66126

1904
A-24

Introduction.

In the following discussion of the principles and applications of single limits, it has been our purpose to test the efficiency of the method of limits as applied to those problems which usually involve infinitesimal quantities. We have also attempted to investigate the extent to which this method may be used in the solution of the problems of mathematics. All those cases, in which single limits may be used in connection with double and simultaneous limits, have been excluded, the idea being to employ the single limit alone.

The general method of development of the subject has been as follows. The

This Book is
UIUC
Illinois Library

first chapter is devoted entirely to a discussion of the principles of single limits. In the first section of this chapter we have given the definitions and geometrical conceptions necessary for a clear understanding of the idea of a limit. Then the properties of the single limit were developed. The purpose was to make the proofs of the existence of these properties of a sufficiently rigorous character that the subsequent discussions might be made to depend directly upon them. The second chapter deals entirely with certain theorems which may be demonstrated by the applications of single limits. The first section is taken up with the discussion of certain miscellaneous theorems. In section two, the question of continuous and discontinuous functions and their properties is treated. The purpose of the third section is to show that the definition and the properties of the definite integral may be based entirely upon limits. The fourth and last section shows that the simple

applications of the integral calculus may be based entirely upon limits, thus discarding the infinitesimal quantities of the various orders.

In this work we have consulted the following authorities

Dini.	"Theorie der Funktionen einer veränderlichen reellen Grösse".
Chrystall	"Algebra"
Serret.	"Differential-und-Integral Rechnung"
Stoltz	"Differential-und-Integral Rechnung."
Youngand Lineburger	"Elements of Calculus"
Murray	"Integrals calculus"
Harnack	Introduction to Calculus

Jordan	"Cours D'Analyse"
Genocchi-Peano	"Differential-und-Integral Rechnung."
Taylor	"Integral Calculus"
Newcomb	"College Algebra"
Prof. E. J. Townsend	"Lectures" in Math 13.a. given 1903-1904.

Contents.

Chapter I.

Principles of Single Limits.

Section.		Page.
1.	Definitions	1.
	Conception of limits	4
	Geometrical conception of limits	4
	Example 4. $y = \sin x$	4.
	Example 5. $y = \sin \frac{1}{x}$	5
	Example 6. $y = x \sin \frac{1}{x}$	6
	Example 7 $y = x^2 \sin \frac{1}{x}$	6
	Existence of a limit right & left of a point	8
2.	Properties of limits	11
	Limit of a constant times a function	12
	Limit of a sum	13
	Limit of a product	15
	Limit of $\frac{f(x)}{\varphi(x)}$	18.

Section.		Page
	Limit of $\frac{\delta}{\sin\delta}$	20.
	Limit of a function the function of the limit	21
	$\underset{x\doteq\infty}{L}\left\{f(x+1)-f(x)\right\}=\underset{x\doteq\infty}{L}\frac{f(x)}{x}$	22.
	$\underset{x\doteq\infty}{L}\frac{f(x+1)}{f(x)}=\underset{x\doteq\infty}{L}\left\{f(x)\right\}^{\frac{1}{x}}$	24.
	$L\left\{f(x)\right\}^{\varphi(x)}=\left\{L f(x)\right\}^{L\varphi(x)}$	25.

Chapter II.

Some Applications of Limits.

1.	Miscellaneous theorems	27.
	Every regular sequence of rational numbers has a limit	27.
	Every regular sequence defines only one number	29.
	Monotone functions have limits	32.
	A number may be expressed in a system having any integral positive number as its base	34
2.	Continuous and discontinuous functions	36.

Section.	Page.
Art I. Continuous and discontinuous functions	36.
Continuous functions defined	36.
Discontinuous functions defined	36.
Upper limit	36
Maximum value	37.
Lower limit	37.
Minimum value	37.
Oscillation of a function	37.
Amount of discontinuity	37.
Continuity for a given interval	38
Uniformity continuity	38.
II. General properties of continuous and discontinuous functions	39.
Constant times a continuous function	39
Algebraic sum of continuous functions	40
Product	41
Quotient	42
$a^{f(x)}$ is continuous if $f(x)$ is continuous	42.
$y = [f(x)]^{\phi(x)}$	43
$\frac{1}{f(x)}$ continuous if $f(x)$ is continuous	43
A continuous function defined for a set of points is completely defined	44.

Section. Page.

. Limit of the oscillations of a continuous function is zero. 45.

A function continuous throughout an interval, end points included, is uniformly continuous 46.

A continuous function reaches its upper limit. 49.

A single valued continuous function, in the interval (α, β) equals zero at least once if $f(\alpha)$ and $f(\beta)$ have opposite signs 51.

A single valued continuous function, in the interval (α, β), passes through all values between $f(\alpha)$ and $f(\beta)$ 54.

If $y = f(x)$ is single valued and continuous, then the function $x = \varphi(y)$ is continuous 57.

If $f(x)$ and $\varphi(x)$ are continuous and if $u = \varphi(x)$, then $y = f(u)$ is continuous 58

If a continuous function takes values which are less than any arbitrary number, in a given interval, then for some definite value of x, the function equals zero 60

Section. Page.

If a function is continuous at a point x' and, at a set of points dense at x', takes the value A, then $f(x') = A$ always. 61.

A set of points at which the (x,y)-discontinuity $\geqq A$, cannot be everywhere dense in any sub-region 62.

The points at which $f(x,y)$ is continuous with respect to both variables, form a set everywhere dense 66.

If a point-wise discontinuous function is constant at a set of points everywhere dense, then the points at which the function is continuous belong to this set 69.

Characteristic of a point-wise discontinuous function 70.

If $f(x)$ has a discontinuity $\geqq A$ at a set of points dense at x_0, then it has such a discontinuity at x_0 itself 73

3. The definite integral 76.

Example 1. $\int_a^b x^m \, dx$. 84.

Section.		Page.
	Necessary and sufficient condition for the existence of a definite integral	87.
	The definite integral expressed as the difference between the indefinite integrals for the upper and lower limits.	90.
	$\int_a^b f(x)\,dx = (b-a) f(\xi_1)$	94.
	Definite integral as a continuous function of its upper limit	95.
	$\int_a^b f(x)dx = \lim_{x \doteq b} F(x) - \lim_{x \doteq a} F(x)$	97.
	Interchanging the order of the limits	98
	$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$; $a < c < b$	99.
	If $f_1(x), f_2(x)$ --- are integrable, then $f_1(x) \pm f_2(x) \pm f_3(x) \pm$ is also integrable	99
	Product of a finite number of integrable functions is integrable	100.
	$\frac{f(x)}{\varphi(x)}$ may be integrated if $f(x)$ and $\varphi(x)$ are both integrable	101.
	$\frac{1}{f(x)}$ is integrable if $f(x)$ can be integrated and has its maximum and minimum values of the same sign	103.
	Special functions which can be integrated	104.

Section. Page.

1. Those continuous in general and point-wise discontinuous 104
2. Those with a finite number of discontinuities > 0 109.
3. Those with ordinary discontinuities or discontinuities of the second kind which exist only on the same side of the points in question 110
4. Those which are totally discontinuous are not integrable 115.

changing a function in an arbitrary assemblage of points of the 1st or 2nd species does not change the integral 116.

Conditions under which $\int_\alpha^\beta f(x)\,dx$ has the same or opposite sign as $f(x)$ has 118.

Functions whose integrals are always equal to zero. 119.

If $f(x)$ is integrable, $|f(x)|$ can be integrated 121.

Relation between $\int_\alpha^\beta f(x)\,dx$ and $\int_\alpha^\beta \varphi(x)\,dx$ for $f(x) \geq \varphi(x)$, according as $\alpha < \beta$, or $\alpha > \beta$ 122.

Relation between the absolute value of the integral of $f(x).\varphi(x)$ and the integral of $|\varphi(x)|$ 123.

Section		Page
	conditions under which $L\int_a^b \varphi(x)\,dx \leq \int_a^b f(x)\varphi(x)\,dx \leq l\int_a^b \varphi(x)\,dx$	124.
4.	Applications of definite integrals	126.
	Areas of curves	126.
	Volumes of solids of revolution	127
	Example. Volume of a cone	131
	Lengths of curves	134
	Areas of surfaces of solids of revolution	137

Chapter I.

Principles of Single Limits.

§1. Definitions.

Limit. If a variable takes successive values that approach nearer and nearer to a given constant, so that the difference between the variable and the constant may become smaller, in absolute value, than any number that can be assigned, then the constant is called the limit of the variable.

y is said to be a function of x, $y = f(x)$, if for every value of x, for which x is defined, there corresponds a definite value of y. The magnitude x may not be defined for every value, but for every value, for which it is defined, there corresponds a definite value for y. In a finite interval, say (a,b), let x assume an infinite number of values $x = \alpha_1, \alpha_2 - - - \alpha_n$, then these x-values must be dense at least at one point of the interval. Call this point x_0. Since there is an infinite number of x values, there must be an infinite number of y values, for which there must be at

least one limiting point. As a matter of fact, these y-values must be dense at only one finite point; they may be dense at several points, or they may be dense at no finite point. Let the x-values be dense at one point. Then the y-values approach as near as we please to a definite finite value. These values may be made to differ as little as we please from this definite number by taking x sufficiently near to the limiting point of the x-values. Or, expressing the same thing symbolically, we write

$$\lim_{x \doteq a} f(x) = A.$$

where A is the definite finite number referred to above, and where the x-values are dense at the point $x = a$.

We have stated above that the function must have one or more points of condensation. We may say then that

$$|f(a+\delta) - A| < \epsilon \quad \text{for } |\delta| < \delta_1$$

where δ_1 is a definite finite number depending upon ϵ and A. i. e. δ_1 is a function of (ϵ, A). ϵ is an arbitrarily small positive number selected in advance.

From the nature of a limit we see at once that all the values of the function in the neighborhood of a point must be taken to show the existence of a limit. That the limit does not depend upon the value of the function at the point, but on the value in the neighborhood of the point in question, may be shown by the following examples.

Ex. 1. Let the value of the function be defined as follows.

$$f(x) = \cos x \qquad \text{for } x \neq 0.$$
$$f(0) = 0 \qquad \text{" } x = 0.$$
$$\lim_{x \doteq 0} \cos x = 1.$$

The value of the limit differs from that of the function at the point, as the value of the latter is zero.

Ex. 2. Let

$$f(x) = \lim_{n \doteq \infty} x^{\frac{2}{n}}.$$

Now if we let x approach zero we have

$$\lim_{x \doteq 0} f(x) = 1$$

which differs from $f(0)$ which is 0.

Ex. 3. Let

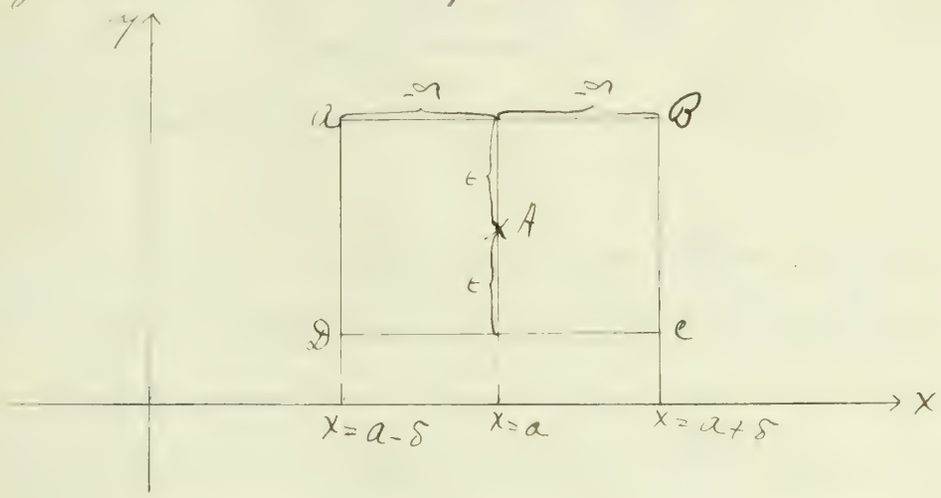
$$y = 0 \qquad \text{for } x = 0$$
$$y = 1 - x \qquad \text{" } x \neq 0.$$

As x approaches zero we have

$$\lim_{x=0} y = \lim_{x=0} (1-x) = 1$$

while by definition, the value of the function for $x=0$, is zero.

"The conception of a limit may be shown geometrically thus.



The value ϵ is chosen arbitrarily and just as small as is desired, but different from zero. Then δ must be so selected that the value of the function, for all values of x within the interval $x-\delta$ and $x+\delta$, shall lie inside the rectangle A B C D. As an illustration let us take the following example.

Ex. 4. $\lim_{x=0} \sin x$

or

$$|f(0+\delta)-0| < \epsilon.$$

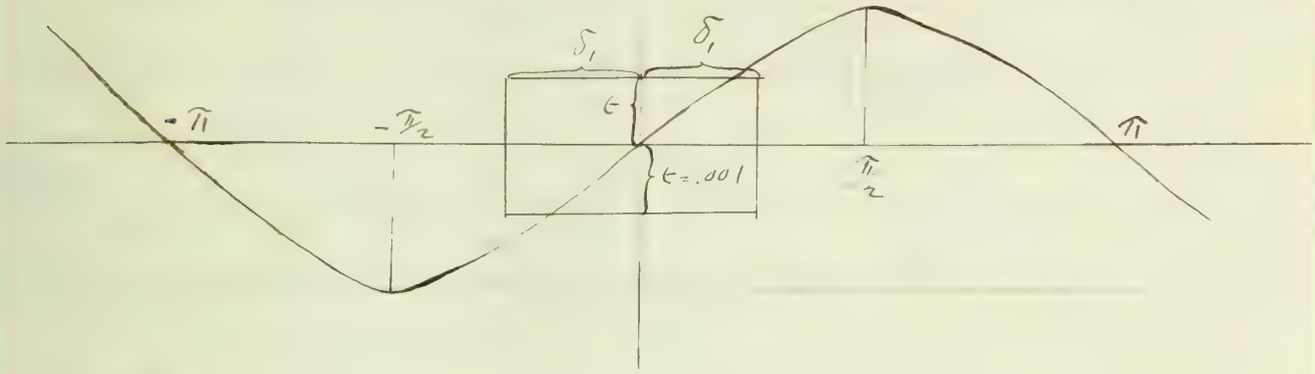
Then

$$f(\delta) = \sin\delta$$
$$|\sin\delta - 0| < \epsilon$$

$\therefore$

$$|\sin\delta| < \epsilon.$$

Geometrically we have



For an oscillating function, the limit does not exist if the oscillations remain greater than some finite value which is itself zero. For example let us consider the following function.

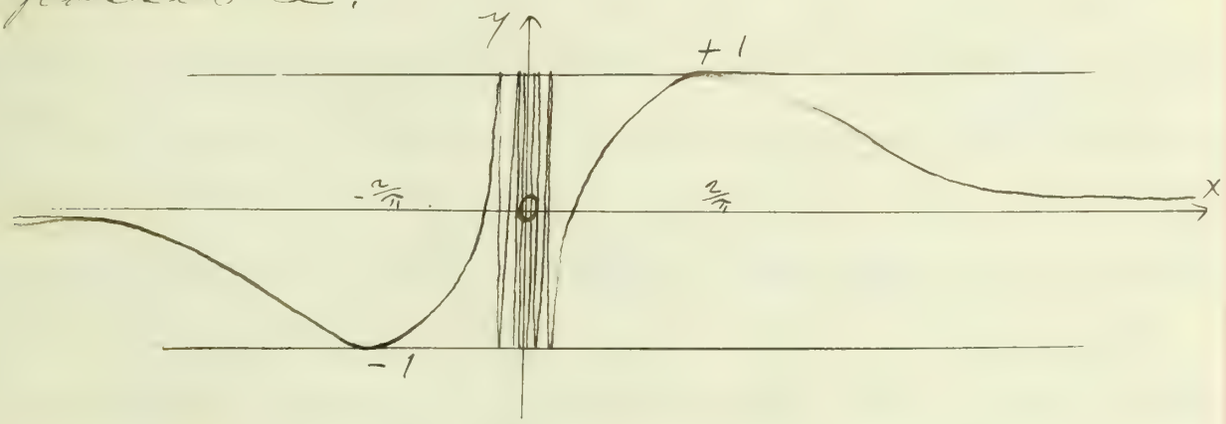
Ex. 5. Let

$$y = f(x) = \sin \frac{1}{x}$$

To prove

$$|\sin \tfrac{1}{\delta} - A| < \epsilon \qquad |\delta| < \delta_1$$

if possible.



The function oscillates back and forth between +1 and -1, the oscillations becom-

ing dense on either side of the origin. From the diagram it is easily seen that the function does not remain within any arbitrarily small rectangle as $x = a$, and hence the limit does not exist. This function may be modified in such a way as to have a limit.

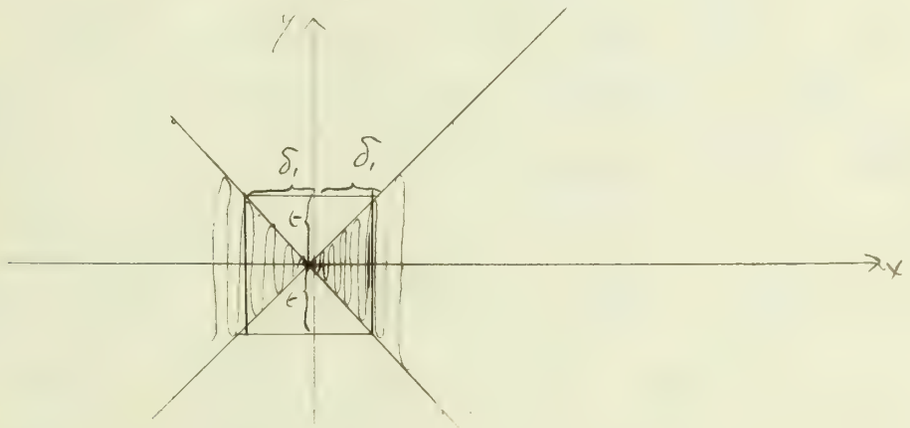
Ex. 6. Let

$$y = f(x) = x \sin \frac{1}{x}$$

Let x approach zero and we have

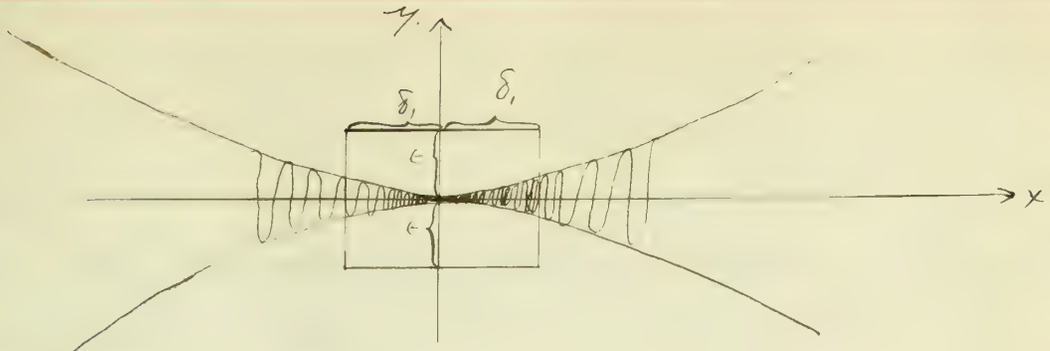
$$|\delta \sin \tfrac{1}{\delta} - 0| < \epsilon \quad \text{for } |\delta| < \delta_1.$$

$$|\delta \sin \tfrac{1}{\delta}| < \epsilon$$



In this problem the function becomes dense at the origin also, but remains between the two 45° lines through the center. Consequently the function remains within the arbitrarily small rectangle about the origin, and the limit exists. Let us examine another function

Ex. 7. $f(x) = x^2 \sin \frac{1}{x}$.



In this problem, the oscillations of the function become dense at the origin, on either side, but always remain between the two curves which are tangent to each other at the origin. As the figure shows, the function remains within the arbitrarily small rectangle about the origin.

The definitions which have been given for a limit apply whether the value of the function is always greater or always less or sometimes greater and sometimes less than the constant. An illustration of the case wherein the value of the function may be sometimes greater and sometimes less than the constant is found in the series

$$1, \quad -\tfrac{1}{2}, \quad +\tfrac{1}{4}, \quad -\tfrac{1}{8} \cdots$$

for an infinite number of terms.

The sum of n terms approaches $\tfrac{2}{3}$ as a limit when n increases without limit. The first term is 1, the sum of

the first two terms $\frac{1}{2}$; of the first three $-\frac{3}{4}$ etc. These sums are alternately greater and less than $\frac{2}{3}$, but any one of them is nearer $\frac{2}{3}$ than any preceding sum. By taking terms enough, a sum can be reached that will differ from $\frac{2}{3}$ by less than any number that may be assigned. The sum of "n" terms is

$$S_n = \frac{2}{3}\left[1 - \left(-\frac{1}{2}\right)^n\right]$$

$$S_n - \frac{2}{3} = -\frac{2}{3}\left(-\frac{1}{2}\right)^n$$

Then

$$\lim_{n=\infty}\left(S_n - \frac{2}{3}\right) = \lim_{n=\infty} -\frac{2}{3}\left(-\frac{1}{2}\right)^n$$

$$= 0$$

Hence we have shown that when n becomes indefinitely large S_n differs from $\frac{2}{3}$ by less than any number.

Sometimes a limit exists to the right of a given point and to the left does not exist and vice versa. We express the limit as a variable approaches a certain value from the right by $\lim_{x=a+}$ and the limit as the variable approaches the value from the left by $\lim_{x=a-}$. As illustrations of these different cases we have the following examples.

Ex. 8. $y = f(x) = \cos\frac{1}{x^2}$ for $x > 0$.

$\therefore \; y' = f'(x) = \cos x^2 \qquad$ for $x \lessgtr 0.$

Then let x approach 0 from the right side and we have

$$\lim_{x \doteq 0+} \cos \frac{1}{x^2} = (?)$$

i.e. does not exist. On the other hand, if we let x approach 0 from the left

$$\lim_{x \doteq 0-} \cos x^2 = 1.$$

In this example the limit exists at the left but not at the right of the point $x = 0$.

Ex. 9. $f(x) = \dfrac{1}{x-a}$

$$\lim_{x \doteq a+} \frac{1}{x-a} = \frac{1}{0} = \infty$$

$$\lim_{x \doteq a-} \frac{1}{x-a} = \lim_{x \doteq a+} -\frac{1}{x-a} = -\infty$$

In this case the limit becomes infinite on both sides, but one is a negative quantity and the other is positive.

Ex 10. $y = x^2 + \sin\left|\frac{1}{x}\right| + \sin\frac{1}{x}$

for $-1 \lessgtr x \lessgtr +1.$ $\qquad y = 0$ for $x = 0.$

Let $x = 0 + \delta$. Then

$$\lim_{y \doteq 0} y = \lim_{\delta \doteq 0} \left[(0+\delta^2) + \sin\left|\frac{1}{0+\delta}\right| + \sin\frac{1}{0+\delta}\right] = ?$$

This limit is then indeterminate. Now let $x = 0 - \delta$, and we have

$$\lim_{y \doteq 0} y = \lim_{\delta \doteq 0} (0-\delta)^2 + \sin\left|\frac{1}{0-\delta}\right| + \sin\frac{1}{0-\delta} = 0.$$

Ex. 11. If $f(x) = \cos \frac{1}{x-a}$ for $x < a$
$= \cos x$ " $x \geq a$.

Now $\lim_{x \doteq a+} \cos \frac{1}{x-a} = (?)$

while

$\lim_{x \doteq a+} \cos x = \cos a.$

Here the limit exists at the right but does not exist at the left.

It should be noted that, by using the absolute values in our definition, we investigate the function on both sides of the point under consideration both right and left; and the definition states that, if the limit for $x=a$ exists and is equal to A, the limit to the right for x approaching a through positive values, is equal to the limit to the left, for x approaching through negative values, and both are equal to A. As we have just seen the limit may exist on one side of a point and not exist on the other, or the limit may not exist at all. However, if we know in advance that the limit exists then we may select any particular set of values dense at the point.

If we have

(1) $\lim_{x \doteq a} f(x) = A$

we may write

(2) $f(x) = A + \alpha$

where α is defined as an arbitrary number, either positive or negative, such that its value approaches zero as x is taken sufficiently near to a. That equation (2) is true may be seen if we write

$$\lim_{n \doteq \infty} f(x) = \lim_{n \doteq \infty} (A + \alpha)$$
$$= A$$

because of the nature of α.

§ 2. Properties of Limits.

Theorem I. When $\lim_{x \doteq a} f(x) = A$, then for any set of values $x = x_1, x_2, x_3 \cdots x_n$ dense at $x = a$ we have $\lim_{x_n \doteq a} f(x_n) = A$.

Since $\lim_{x \doteq a} f(x) = A$ we may write.

$$f(x) = A + \alpha$$

where α is an arbitrary number, either positive or negative, such that its value approaches zero as x is taken sufficiently near to a. Since x takes the values $x_1, x_2 \cdots x_n$ we have

$$f(x_1) = A + \alpha'$$
$$f(x_2) = A + \frac{\alpha'}{2}$$
$$f(x_3) = A + \frac{\alpha'}{3}$$
$$\cdots \quad \cdots \quad \cdots$$
$$f(x_n) = A + \frac{\alpha'}{n}$$

where α' is defined as α was defined above. As n becomes infinitely large we have x_n approaching a. Then

$$\lim_{x_n \doteq a} f(x_n) = \lim_{x_n \doteq a} \left(A + \frac{\alpha'}{n}\right)$$
$$= A.$$

and the theorem is proved.

It should be noted that the converse of this theorem is not true.

Theorem II. The limit of a constant times a function is equal to the constant times the limit of the function, provided the limit exists. i. e.

$$\lim_{x \doteq a} c\, f(x) = c \lim_{x \doteq a} f(x)$$

if

$$\lim_{x \doteq a} f(x) = A.$$

Proof. We have seen that we may write $f(x) = A + \alpha$ where α is some arbitrarily small function of x and A is the limit of the original function of x. Then

$$c\, f(x) = c[A + \alpha]$$

$$\lim_{x \doteq a} c f(x) = c A = c \lim_{x \doteq a} f(x)$$

because as x approaches a, as near as we please, the arbitrarily small function α becomes as small as we please and in the limit becomes zero. Hence the theorem

Theorem III. When the limit of $f(x)$ exists as x approaches a and the limit of $\phi(x)$ also exists as x approaches a, then

$$\lim_{x \doteq a} \{ f(x) \pm \phi(x) \} = \lim_{x \doteq a} f(x) \pm \lim_{x \doteq a} \phi(x).$$

Proof. As above we have

(1) $$f(x) = A + \alpha$$

(2) $$\phi(x) = B + \beta$$

where α and β can be made as small as we choose by taking x sufficiently near to the value $x = a$. Adding equations (1) and (2) we have

$$f(x) + \phi(x) = A + B + (\alpha + \beta).$$

where again we can make $(\alpha + \beta)$ as small as we choose by the proper selection of x. Then

$$\lim_{x \doteq a} \{ f(x) + \phi(x) \} = A + B = \lim_{x \doteq a} f(x) + \lim_{x \doteq a} \phi(x).$$

as the theorem demands.

The converse of this theorem is however not true. As an illustration of cases when this converse is not true let us

examine the following examples.

Ex. 1. $f(x) = \frac{a}{x}$, $\phi(x) = \frac{bx-a}{x}$

$\lim_{x \doteq 0} f(x)$ does not exist.

$\lim_{x \doteq 0} \phi(x)$ " " "

But

$$\lim_{x \doteq 0} \{ f(x) + \phi(x) \} = \lim_{x \doteq 0} \left\{ \frac{a}{x} + \frac{bx-a}{x} \right\} = \lim_{x \doteq 0} \frac{bx}{x} = b.$$

Note: The limit of the sum of an infinite number of functions may not be equal to the sum of their limits.

Ex. 2. When $n \doteq \infty$ the sum of the limits of

$$\frac{1}{2} + \frac{1}{n^2}, \frac{1}{2^2} + \frac{2}{n^2}, \frac{1}{2^3} + \frac{3}{n^2} \text{ ---- } \frac{1}{2^n} + \frac{n+1}{n^2}$$

is 1 but the limit of their sum is $\frac{1}{2}$.

Ex. 3. $\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \text{---} = \frac{x}{2}$

The limit of the sum for $x = \pi +$ is $+\frac{1}{2}\pi$, and the limit of the sum for $x = \pi -$ is $-\frac{1}{2}\pi$. But the sum of the limits is zero. We may state under this theorem the following

Corollary. If $\lim_{x \doteq a} \{ f(x) + \phi(x) \} = A$, and $\lim_{x \doteq a} f(x)$ exists, then $\lim_{x \doteq a} \phi(x)$ must exist also.

Proof: Let $\lim_{x \doteq a} f(x) = B$, then

(1) $$f(x) + \phi(x) = A \pm \alpha$$

(2) $$\psi(x) = B \mp \beta$$

Subtracting (2) from (1) we have

(3) $$\phi(x) = (A - B) \pm (\alpha \mp \beta)$$

where α and β become indefinitely small, the nearer x approaches a. Hence

$$\lim_{x \doteq a} \phi(x) = A - B.$$

This is some definite quantity and we have proved that the limit of $\phi(x)$ exists as x approaches a.

Theorem IV. *If $\lim_{x \doteq a} \phi(x)$ and $\lim_{x \doteq a} \psi(x)$ both exist, then*

$$\lim_{x \doteq a} \{ \phi(x) . \psi(x) \} = \lim_{x \doteq a} \phi(x) . \lim_{x \doteq a} \psi(x).$$

Proof. Since $\lim_{x \doteq a} \phi(x)$ and $\lim_{x \doteq a} \psi(x)$ both exist we may write

(1) $$\phi(x) = A + \alpha$$

(2) $$\psi(x) = B + \beta$$

where α and β may be made as small as we please by allowing x to approach sufficiently near to a. Then multiplying (1) and (2) together we have

(3) $$\phi(x) . \psi(x) = AB + (\alpha\beta + \beta A + \alpha B).$$

then

$$\lim_{x \doteq a} \phi(x) . \psi(x) = AB = \lim_{x \doteq a} \phi(x) . \lim_{x \doteq a} \psi(x).$$

The converse of the above theorem is

however not true; that is, $\lim_{x \doteq a} \{\phi(x) \cdot \psi(x)\}$ may exist without both $\lim_{x \doteq a} \phi(x)$ and $\lim_{x \doteq a} \psi(x)$ existing. This is shown by the following example.

Ex. 1. $\lim_{x \doteq a} [(x-a) \sin \frac{1}{x-a}] = 0$

But

$\lim_{x \doteq a} \sin \frac{1}{x-a}$ does not exist.

On the otherhand

$$\lim_{x \doteq a} (x-a) = 0.$$

Ex. 2. Let $y = f(x) = \frac{3x^4 - 4x^3}{x^3}$.

$$\lim_{x \doteq \infty} \frac{3x^4 - 4x^3}{x^3} = \lim_{x \doteq \infty} (3x - 4) = \infty$$

and

$$\lim_{x \doteq \infty} \frac{1}{x} = 0.$$

But

$$\lim_{x \doteq \infty} \left[\frac{3x^4 - 4x^3}{x^3} \cdot \frac{1}{x} \right] = \lim_{n \doteq \infty} \left(3 - \frac{4}{x}\right) = 3.$$

Ex. (3) $\lim_{x \doteq \infty} \frac{1}{x} = 0$, $\lim_{x \doteq \infty} x = \infty$.

But

$$\lim_{x \doteq \infty} \frac{1}{x} \cdot x = 1.$$

The limit of the product of an infinite number of terms is not the product of the limit of the terms. Let us consider as an illustration the following example.

Ex (4). $\lim_{n \doteq \infty} a^{\frac{1}{n}} = a^0 = 1$

$\lim_{n \doteq \infty} \{a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot \text{- - - - - - } n \text{ terms}\} = \lim_{n \doteq \infty} a^{\frac{n}{n}} = a.$

Hence

$$\lim_{n=\infty} a^{\frac{1}{n}} \cdot \lim_{n=\infty} a^{\frac{1}{n}} \text{ ---- to } n \text{ terms} = 1.$$

which is different from the limit of the product which is a.

From this theorem we can formulate the following corollaries.

Corollary I. If $\lim_{x=a} [\phi(x) \cdot f(x)] = A$ and if $\lim_{x=a} f(x)$ exists and is different from zero, then $\lim_{x=a} \phi(x)$ must exist also.

Let $\lim_{x=a} f(x) = B \qquad$ for $B \neq 0$.

Then

$$(1) \qquad f(x) \cdot \phi(x) = A + \alpha$$

$$(2) \qquad f(x) = B + \beta$$

where α and β are defined as in the preceding theorems Divide (1) by (2) and we have

$$(3) \qquad \phi(x) = \frac{A+\alpha}{B+\beta} = \frac{A}{B+\beta} + \frac{\alpha}{B+\beta}.$$

Since α and β may be made as small as we please by chosing x sufficiently near to a, by taking the limit of both sides of equation (3) we have

$$\lim_{x=a} \phi(x) = \frac{A}{B}.$$

$\frac{A}{B}$ is a definite finite number, and it follows that the limit exists.

Corollary II. If $\lim_{x \doteq a} f(x) = A$, then for a finite n

$$\lim_{x \doteq a} \{ f(x) \}^n = \{ \lim_{x \doteq a} f(x) \}^n .$$

Theorem V. When $\lim_{x \doteq a} \phi(x)$ and $\lim_{x \doteq a} \psi(x)$ both exist and $\lim_{x \doteq a} \psi(x) \neq 0$, then

$$\lim_{x \doteq a} \frac{\phi(x)}{\psi(x)} = \frac{\lim_{x \doteq a} \phi(x)}{\lim_{x \doteq a} \psi(x)} .$$

Proof. Since $\lim_{x \doteq a} \phi(x)$ and $\lim_{x \doteq a} \psi(x)$ both exist we may write

(1) $$\phi(x) = A + \alpha .$$

(2) $$\psi(x) = B + \beta .$$

Divide (1) by (2) and we have

$$\frac{\phi(x)}{\psi(x)} = \frac{A + \alpha}{B + \beta} = \frac{A}{B + \beta} + \frac{\alpha}{B + \beta} .$$

Since α and β are so chosen that they become as small as we please by letting x approach sufficiently near to a, then

$$\lim_{x \doteq a} \frac{\phi(x)}{\psi(x)} = \frac{A}{B} = \frac{\lim_{x \doteq a} \phi(x)}{\lim_{x \doteq a} \psi(x)} .$$

and our theorem is proved.

The converse of the above theorem is however not true. For illustration we use the following examples.

Ex. 1. $\phi(x) = \frac{1}{x - a}$.

$$\psi(x) = \frac{1}{(x-a)^2}$$

$$\lim_{x \doteq a} \frac{1}{x-a} = \infty \quad , \quad \lim_{x \doteq a} \frac{1}{(x-a)^2} = \infty$$

But

$$\lim_{x \doteq a} \left\{ \frac{\frac{1}{x-a}}{\frac{1}{(x-a)^2}} \right\} = \lim_{x \doteq a} (x-a) = 0.$$

We may have $\lim_{x \doteq a} F(x)$ not existing and $\lim_{x \doteq a} g(x)$ not existing and yet the limit $\lim_{x \doteq a} \frac{F(x)}{g(x)}$ may not exist, as shown by the following example.

Ex. 2. Let

$$F(x) = x^m + a_1 x^{m-1} + \qquad + a_m$$

$$g(x) = x^m + b_1 x^{m-1} + \quad - \quad - + b_m$$

Then

$$\lim_{x \doteq \infty} F(x) = \infty \quad , \quad \lim_{x \doteq \infty} g(x) = \infty$$

but

$$\lim_{x \doteq \infty} \left[F(x) \div g(x) \right] = +\infty, \ 1, \ 0, \text{ as } m \gtrless n.$$

(1) $$F(x) = x^m \left\{ 1 + \frac{a_1}{x} + \quad - \quad - + \frac{a_m}{x^m} \right\}$$

(2) $$g(x) = x^n \left\{ 1 + \frac{b_1}{x} + \qquad + \frac{b_n}{x^n} \right\}$$

Dividing (1) by (2) we have

(3) $$\frac{F(x)}{g(x)} = \frac{x^m \left\{ 1 + \frac{a_1}{x} + \quad - \quad - + \frac{a_m}{x^m} \right\}}{x^n \left\{ 1 + \frac{b_1}{x} + \quad - \qquad + \frac{b_n}{x^n} \right\}}$$

Then taking limits of both sides

$$\lim_{n \doteq \infty} \frac{F(x)}{g(x)} = \lim_{n \doteq \infty} \frac{x^m \left(1 + \frac{a_1}{x} + \quad - \quad - + \frac{a_m}{x^m} \right\}}{x^n \left(1 + \frac{b_1}{x} + \quad - \quad - \quad - + \frac{b_n}{x^n} \right\}} = 1 \text{ for } m = n.$$

and 0 for $m < n$, while D is the value for $m > n$.

Theorem VI. *If we have the functions $\phi(x)$ and $\psi(x)$ such that for all values of x, where $a < x < a+d$ (for $d > 0$) we have*

$$\phi(x) < F(x) < \psi(x);$$

and if $\lim_{x=a} \phi(x) = B = \lim_{x=a} \psi(x)$ where B is a finite number, then $\lim_{x=a} F(x) = B$.

Proof: We have

$$\phi(x) < F(x) < \psi(x).$$

Subtract $\phi(x)$ from both sides. We have

$$0 < F(x) - \phi(x) < \psi(x) - \phi(x).$$

Taking limits of the inequalities

$$0 \leq \lim \{F(x) - \phi(x)\} \leq \{\psi(x) - \phi(x)\}$$

But

$$\lim \{\psi(x) - \phi(x)\} = \lim \psi(x) - \lim \phi(x) = 0.$$

because in our hypothesis, the limits of $\psi(x)$ and $\phi(x)$ were both equal to the same quantity B. Then

$$\lim \{F(x) - \phi(x)\} = 0$$

and

$$\lim F(x) - \lim \phi(x) = 0.$$

Or

$$\lim F(x) = \lim \phi(x) = \lim \psi(x) = B.$$

Hence the theorem.

Ex. We have for δ in the neighborhood

If θ is zero

$$\sin\theta < \theta < \tan\theta$$

or

$$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta}.$$

Dividing by $\sin\theta$ we have

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

But

$$\lim_{\theta \doteq 0} 1 = \lim_{\theta \doteq 0} \frac{1}{\cos\theta} = 1.$$

Hence by our theorem we have

$$\lim_{\theta \doteq 0} \frac{\theta}{\sin\theta} = \lim_{\theta \doteq 0} \frac{1}{\cos\theta} = 1.$$

Theorem VII. *When $\lim_{x \doteq a} f(x)$ and $\lim_{x \doteq a} \varphi(x)$ both exist and $f(x) = \varphi(x)$, for all values in the neighborhood of a, then*

$$\lim_{x \doteq a} f(x) = \lim_{x \doteq a} \varphi(x).$$

Proof: Let us assume that

$$F(x) = f(x) - \varphi(x).$$

Then since $F(x) = 0$, $\lim_{x \doteq a} F(x) = 0$. Then

$$\lim_{x \doteq a} F(x) = \lim_{x \doteq a} \{ f(x) - \varphi(x) \}.$$

Or

$$0 = \lim_{x \doteq a} \{ f(x) - \varphi(x) \} = \lim_{x \doteq a} f(x) - \lim_{x \doteq a} \varphi(x).$$

Therefore

$$\lim_{x \doteq a} f(x) = \lim_{x \doteq a} \varphi(x)$$

and our theorem is proved.

Theorem VIII. *If $\lim_{y \doteq b} F(y) = F(b)$, where y is a function of x such that $\lim_{x \doteq a} y(x) = b$,*

then $\lim_{x \doteq a} F(y) = F(\lim_{x \doteq a} y(x))$.

By hypothesis we have

$$\lim_{y \doteq b} F(y) = F(b) = F(\lim_{x \doteq a} y(x)).$$

Since, when we take the limit of $F(y)$ as $y \doteq b$, y becomes a function of x, and since $\lim_{x \doteq a} y(x) = b$, then as $y \doteq b$, $x \doteq a$. We have then

$$\lim_{x \doteq a} F(y(x)) = F(\lim_{x \doteq a} y(x)),$$

and the theorem is proved.

Ex. 1. If $\lim_{x \doteq a} f(x)$ exists and is equal to A, where $A \neq 0$, then

$$\log \lim_{x \doteq a} f(x) = \lim_{x \doteq a} \log f(x).$$

Ex. 2. Let $F(x) = x^2$ where $x = \log y$. Then $\lim_{y \doteq a} F(x) = \lim_{y \doteq a} \log^2 y = \left\{ \log \lim_{y \doteq a} y \right\}^2$

$$= \log^2 a.$$

Ex. 3. $\lim_{x \doteq 1} \log \left\{ (x^2-1)/(x-1) \right\} = \log \left\{ \lim_{x \doteq 1} \frac{(x^2-1)}{(x-1)} \right\}$

$$= \log 2.$$

Theorem IX. $\lim_{x \doteq \infty} \left\{ f(x+1) - f(x) \right\} = \lim_{x \doteq \infty} \frac{f(x)}{x}$

provided $\lim_{x \doteq \infty} \left\{ f(x+1) - f(x) \right\}$ exists.

Proof. Since x is to be made as large as we please we may put

$$x = n + h.$$

where h is a number not necessarily an integer, but as large as we please and n is an integer, as large as we please. First suppose

$$\lim_{x=\infty} \{ f(x+1) - f(x) \} = K$$

where K is some finite quantity. Since this is true, we can always choose for h a definite value so large that for $x = h$ and all greater values

$$| f(h+1) - f(h) - K | < \epsilon$$

no matter how small ϵ may be. Hence we have

$$| f(h+2) - f(h+1) - K | < \epsilon$$
$$| f(h+3) - f(h+2) - K | < \epsilon$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$| f(h+n) - f(h+n-1) - K | < \epsilon$$

and by addition

$$f(h+n) - f(h) - nK < n\epsilon,$$

that is

$$f(x) - f(h) - (x-h) K < (x-h) \epsilon.$$

Hence

$$\frac{f(x)}{x} - \frac{f(h)}{x} - \left(1 - \frac{h}{x}\right) K < \left(1 - \frac{h}{x}\right) \epsilon$$

$$\frac{f(x)}{x} < K + \epsilon + \frac{f(h)}{x} - \frac{h(K+\epsilon)}{x}.$$

We can make h definite because as n increases indefinitely by integral values, the points become dense at α. The limit of such an assemblage is the same as if x were given all values. Therefore since $f(h)$, α, h and k are in the present fixed the result is, that by making x sufficiently large we can make $\frac{f(x)}{x} - k$ numerically less than ϵ. Now ϵ can be made as small as we please by properly choosing h; hence the theorem follows.

Theorem X. *If the limit* $\lim_{x=\infty} \frac{f(x+1)}{f(x)}$ *exists, we have*

$$\lim_{x=\infty} \frac{f(x+1)}{f(x)} = \lim_{x=\infty} \left\{ f(x) \right\}^{\frac{1}{x}}.$$

Proof. From the previous theorem we have

$$\lim_{x=\infty} \left\{ \psi(x+1) - \psi(x) \right\} = \lim_{x=\infty} \frac{\psi(x)}{x}$$

where $\psi(x)$ is any function such that

$$\lim_{x=\infty} \left\{ \psi(x+1) - \psi(x) \right\}$$

exists. Let

$$\psi(x) = \log f(x);$$

so that

$$\psi(x+1) - \psi(x) = \log f(x+1) - \log f(x).$$
$$= \log \left\{ \frac{f(x+1)}{f(x)} \right\}$$

and

$$\frac{\psi(x)}{x} = \left\{ \frac{\log f(x)}{x} \right\} = \log \left\{ f(x) \right\}^{\frac{1}{x}}.$$

Then we have

$$\lim_{x=\infty} \log \left\{ \frac{f(x+1)}{f(x)} \right\} = \lim_{x=\infty} \log \left\{ f(x) \right\}^{\frac{1}{x}}$$

Hence

$$\log \left\{ \lim_{x=\infty} \frac{f(x+1)}{f(x)} \right\} = \log \left[\lim_{x=\infty} \left\{ f(x) \right\}^{\frac{1}{x}} \right].$$

provided

$$\lim_{x=\infty} \frac{f(x+1)}{f(x)} \text{ exists.}$$

Hence finally

$$\lim_{x=\infty} \frac{f(x+1)}{f(x)} = \lim_{x=\infty} \left\{ f(x) \right\}^{\frac{1}{x}}.$$

Theorem XI. *For any critical value of* x

$$\lim \left\{ f(x) \right\}^{\phi(x)} = \left\{ \lim f(x) \right\}^{\lim \phi(x)}$$

provided

$$\lim \left\{ f(x) \right\}^{\lim \phi(x)} \text{ exists.}$$

The only question now is as to the existence of the function of the limits. We may write

$$\left\{ f(x) \right\}^{\phi(x)} = e^{\phi(x) \log f(x)}.$$

Now the limit of $w = \log u$ exists so long as w is finite and equal to or greater than 1. The limit of e^{vw} also exists when v and w have any finite value whatever. Hence so long as $\lim \phi(x)$ and $\lim \log f(x)$ exists we have

$$\angle\{f(x)\}^{\phi(x)} = \angle e^{\phi(x)\log f(x)}$$
$$= e^{\angle\phi(x)\angle\log f(x)}$$
$$= e^{\angle\phi(x).\log\angle f(x)}$$

Hence

$$\angle\{f(x)\}^{\phi(x)} = \{\angle f(x)\}^{\angle\phi(x)}.$$

Chapter II.

Some Applications of the Method of Limits.

§1. Miscellaneous Theorems Involving Limits

Theorem I. Every regular sequence of rational numbers has a limit.

If the sequence $(\alpha) = (\alpha_1, \alpha_2, \alpha_3 \text{ ----})$ is a regular sequence, from some point on, the terms are either all positive or all negative. Call this point α_n. We shall assume that we have a sequence

1.) $(\alpha) = \alpha_n, \alpha_{n+1}, \alpha_{n+2} \text{ ----}$

where all of the terms are greater than zero, or that we have a sequence

2.) $(\beta) = \beta_n, \beta_{n+1}, \beta_{n+2} \text{ ------}$

where all of the terms are negative.

Assuming that (1) is true, we have either of two things happening.

A.) From some point on all the terms continually increase or continually decrease.

B. There is no such point.

Let us consider A.) Each of the α's is less

than some finite number N, i.e.

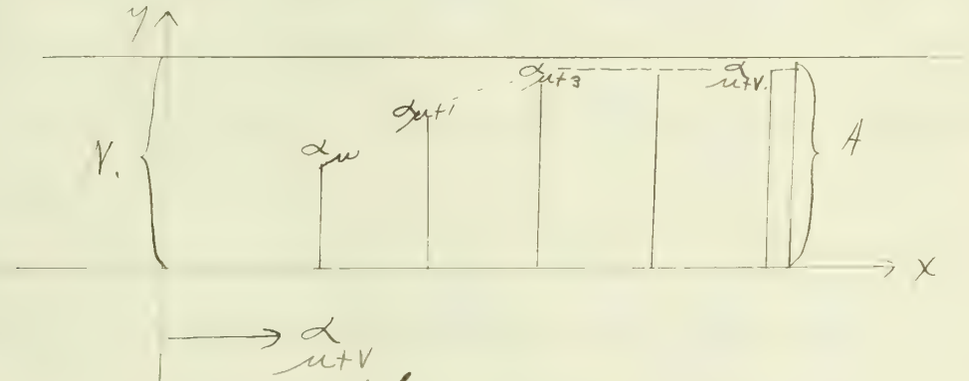
$\alpha_n < \alpha_{n+1} < \alpha_{n+2}$ ---------- each being less than N. Now let us select a set of finite numbers $\epsilon_0\ \epsilon_1\ \epsilon_2 - - - - \epsilon_\nu$ such that

$$\lim_{\nu=\infty} \epsilon_\nu = 0,$$

and let

$$|A - \alpha_n| < \epsilon_0 \quad , \quad |A - \alpha_{n+1}| < \epsilon_1 \quad - -$$
$$|A - \alpha_{n+\nu}| < \epsilon_\nu \quad ; \nu = 1, 2, 3, \ldots$$

where $A \leq N$.



Then we may write

$$\left| \lim_{\nu=\infty} A - \lim_{\nu=\infty} \alpha_{n+\nu} \right| < \lim_{\nu=\infty} \epsilon_\nu$$

$$\therefore |A - \lim_{\nu=\infty} \alpha_{n+\nu}| = 0,$$

and

$$\lim_{\nu=\infty} \alpha_{n+\nu} = A.$$

Then

$$\lim (\alpha) = A.$$

and our theorem is proved for case (A). In the same way we can prove the theorem for the case when the terms are all decreasing from some point on.

Now we shall consider case (B). We have the condition that

$$\alpha_u < \alpha_{u+1} < \alpha_{u+2} \quad < \alpha_{u+v} > \alpha_{u+v+1} > \alpha_{u+v'}$$

The α's oscillate back and forth. In this case either of two things will happen.

A'). There is no term α_v such that it is larger than all the other following terms. i.e. there is no α_v such that $\alpha_v > \alpha_{v+1}$

B.) Or there is such a term.

Let us first consider A'. This is a regular sequence,

$$(\alpha_v) = (\alpha_v, \alpha_{v_1}, \alpha_{v_2} \text{ - - - - - - - - - - })$$

Then by (A) it must have a limit. i. e.

$$\lim_{v=\infty} (\alpha_v) = A.$$

If there is a term greater than all the others then

$$\alpha_v > \alpha_{v_1} > \alpha_{v_2} > \text{- - - - - - - - - - } > \alpha_{v_n},$$

This is a regular, continuously decreasing sequence. Then every regular sequence has a limit.

Theorem II. Every regular sequence of rational numbers defines one and only one number A. This number is the limit of the sequence. It may be expressed by

$$L(\alpha) = A,$$

where $(\alpha) = (\alpha_1 \alpha_2 \cdots\cdots \alpha_K \cdots\cdots)$.

First of all we shall define what we mean by equality of sequence. Let

$$A = L(\alpha) = L(\alpha_1, \alpha_2, \alpha_3 \cdots\cdots \alpha_K \cdots)$$
$$B = L(\beta) = L(\beta_1, \beta_2, \beta_3 \cdots\cdots \beta_K \cdots).$$

and also let

$$|\alpha_1 - \beta_1| < \epsilon_1, \quad |\alpha_2 - \beta_2| < \epsilon_2 \cdots |\alpha_K - \beta_K| < \epsilon_K$$

where $\epsilon_1, \epsilon_2 \epsilon_3 \cdots \epsilon_K \cdots$ is a set of arbitrarily chosen numbers such that

$$\lim_{K=\infty} \epsilon_K = 0.$$

Then we may write

$$\left| \lim_{K=\infty} \alpha_K - \lim_{K=\infty} \beta_K \right| < \lim_{K=\infty} \epsilon_K.$$
$$= 0$$

When the above condition exists we say that the two sequences are equal or that $A = B$.

Having defined equality, we shall now consider irrational numbers, which are defined as being all those numbers which are not rational.

Take all rational numbers and divide them into two groups having the following properties.

1. Every rational number belongs to one

group or the other.

2. Each number A of the first group shall be greater than every number a of the second group. i.e $A > a$.

3. There shall be no smallest number of the first group.

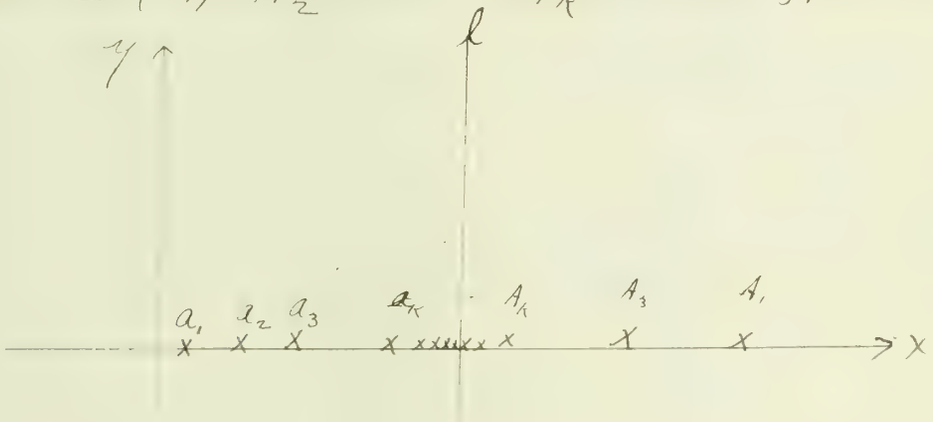
4. There shall be no largest number of the second group.

If we consider these numbers as arranged in order of their magnitude, then each group will have the properties of a regular sequence. Since these two groups of numbers form regular sequences they each have a limit, i.e.

$$L(a) = A, \quad L(A) = B,$$

where $(a) \equiv (a_1, a_2 - - - - a_K - - - -)$

$(A) \equiv (A_1, A_2 \quad\quad A_K \quad\quad)$.



Here each A is larger than a and each a is less than any A. As the two sequences approaches zero, and, according to our definition

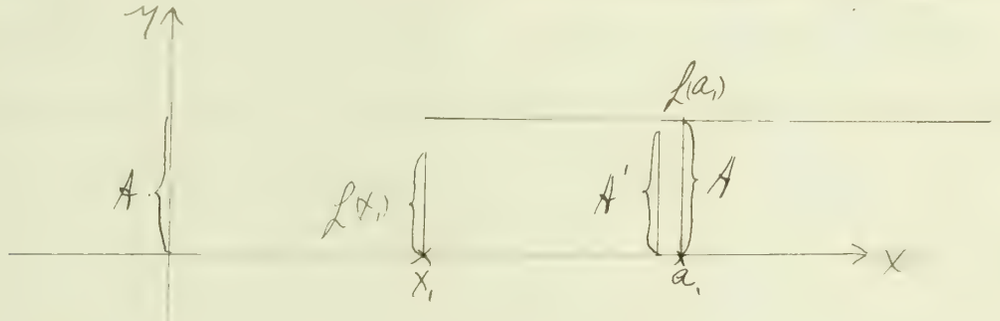
of equality, they are equal. Hence each is an infinite sequence defining one and the same number as the other. This number is represented by the limit and is not a rational number, since all the rational numbers belong to one of the two classes defined as above. Consequently this limiting number is an irrational number.

Theorem III. Every single valued monotone function, whose absolute value is always less than some definite finite number, has a limit.

One of two things happens as x increases. Either

1) Some point x_1, after which the function remains constant as x increases.

2) Or we do not come to such a point.



Suppose $f(x) = c$ from x_1 on. Then as x increases and approaches any point to the right of x_1,

$$C = f(x) = A.$$

where A is the limit of the values of the function as $x = a$. We know that A is the limit of the function, because, by hypothesis, the function is either always increasing or decreasing, up to a certain point, and from there on remains constant. In the one case, the function can never be greater than some definite number, while in the other case it can never be less than some number. Hence, the value of the function, at the point at which the value becomes constant, is necessarily the limit of the function. In case (2) $f(x)$ is always increasing or is always decreasing. If we have the point a, at which the value of the function is equal to A, which is the largest value the function has in a given interval, and if we take values of x as near to a as we please, then we have

$$\lim_{x=a} f(x) = A.$$

Hence in either case there is a limiting value of the function.

Theorem IV. A number may be

expressed in a system having any positive integral number as its base.

Let A be any number which we desire to represent in a system having an arbitrary base b. Suppose we take any number which we may choose as a unit. Lay off from the origin intervals on OA with this unit as a measure. Suppose we have a_0 whole units. Divide this unit into b equal parts and lay off one of these new units on a_0 A. Suppose we have a_1 of these. Divide this unit again into b equal parts and apply this new unit as before on the remaining portion of the interval OA. In this way we build up a number which may be expressed by

$$\phi_n = a_0 + \frac{a_1}{b} + \frac{a_2}{b^2} + \frac{a_3}{b^3} + \text{---} \text{- - -} + \frac{a_n}{b^n}.$$

ϕ_n is a regular sequence and hence has a limit. Since b is larger than 1 we have

$$\lim_{n=\infty} \frac{1}{b^n} = 0.$$

Now by every subdivision of the interval 0 A we reach a number which differs from A by $\frac{1}{b^n}$. We have seen that as the number of subdivisions increases indefinitely, this value of $\frac{1}{b^n}$ becomes indefinitely small. Hence we have

$$\begin{aligned} \lim_{n=\infty} \Phi_n &= \lim_{n=\infty} \left(A - \frac{1}{b^n}\right) \\ &= \lim_{n=\infty} A - \lim_{n=\infty} \frac{1}{b^n} \\ &= A - 0 \\ &= A. \end{aligned}$$

Therefore, since A is any number and b is a positive integral number and the base of a system of numbers, the theorem is demonstrated.

§2. Continuous and Discontinuous Functions.

I. General Definitions.

A function is continuous at a point $x=a$ provided the following condition exist, viz.

$$\lim_{x=a} f(x) = f(a).$$

Two facts are involved in the definition given above. First, it is required that the limit, $\lim_{x=a} f(x)$, must exist and second, that this limit must equal the value of the function at the point in question.

Whenever either of these conditions does not exist, then we say the function is discontinuous. i.e. when

$$\lim_{x=a} f(x) \neq f(a)$$

or

$$\lim_{x=a} f(x) \text{ does not exist.}$$

The upper limit of a function, within a given interval, is that number from which we may make the value of the function to differ as little as we please by a proper choice of the argument x. This number must be such

that the value of the function may never be greater.

When this number is a value of the function itself, it is called the maximum value of the function.

The lower limit of the function is that number from which we may make the function to differ by as little as we please, by a proper choice of the argument x. This number must be such that the function may never be less.

When this number becomes the value of the function itself, it is called the minimum value.

In many discussions it will be necessary to take into account the amount or degree of discontinuity. We may define this as follows. Take some interval (α_n, β_n) which contains the point $x=a$. The function will have variations in the interval. The oscillation of a function in a given interval is the absolute value of the difference between the upper and lower limits of the function within this interval. Then we may define the amount or degree

of discontinuity as follows. If (α_n, β_n) becomes infinitely small as $n = \infty$ then

$$\lim_{n=\infty} (\text{Oscillation}) = \text{Amount or Degree of Discontinuity.}$$

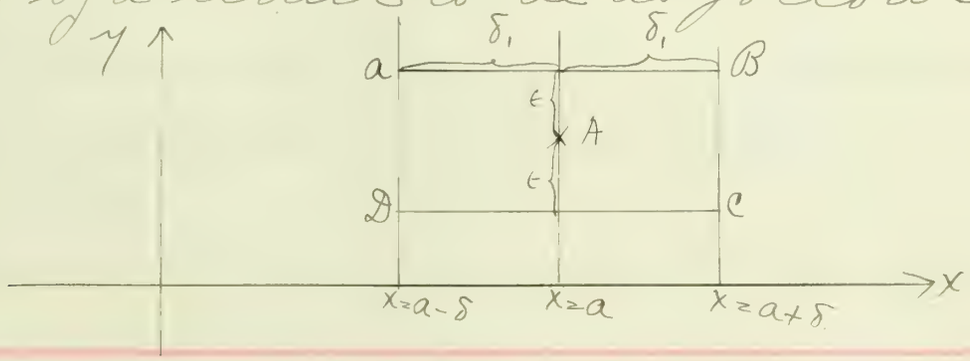
So far we have been speaking of the continuity or discontinuity of a function at a point. Now we shall define continuity or discontinuity throughout an interval.

We say that $f(x)$ is continuous in the interval (α, β) provided we have the following condition

$$\lim_{x=a} f(x) = f(a).$$

existing for every point in the interval where $\alpha \leq a \leq \beta$. The interval may or may not be closed.

We shall now take up the question of uniform continuity. In §1, Chapter I, we have shown the geometrical conception of a limit to be as follows.



The value ϵ is chosen arbitrarily and just as small as is desired, but diffferent from zero. Then a δ must be so selected that the value of the function for all values of x within the interval $(x-\delta, x+\delta)$ shall lie inside the rectangle ABCD. Now if we choose an arbitrarily small ϵ which is to be constant throughout the discussion of the problem, then δ becomes a function of x alone. We should then have as many δ's as we have values of x. Hence there is an assemblage of δ's depending upon the assemblage of x's. There must be a lower limit of these δ's, which may or may not be zero. If for all possible δ's there exists a lower limit different from zero, then we can choose this lower limit as the δ for all the points of the interval. When such a condition exists we have uniform continuity.

II. General Properties of Continuous and Discontinuous Functions.

Theorem I. Any constant $\pm$ times a

continuous function is a continuous function.

If $f(x)$ is continuous at $x = a$ we have
$$\underset{x \doteq a}{L} f(x) = f(a).$$
Then
$$\underset{x \doteq a}{L}\, c\, f(x) = c \underset{x \doteq a}{L} f(x) = c.\ f(a).$$
and the theorem is proved.

Theorem II. The algebraic sum of a finite number of continuous functions is a continuous function.

Let
$$f(x) = \phi_1(x) + \phi_2(x) + \phi_3(x) + \text{-----} + \phi_n(x).$$
where all of these functions are continuous. Then
$$\underset{x \doteq a}{L} \phi_k(x) = \phi_k(a); \quad k = 1, \text{----} n.$$
Then
$$1)\ \underset{x \doteq a}{L} f(x) = \underset{x \doteq a}{L} \left\{ \phi_1(x) + \phi_2(x) + \text{-----} + \phi_n(x) \right\}$$
$$= \underset{x \doteq a}{L} \phi_1(x) + \underset{x \doteq a}{L} \phi_2(x) + \text{----} + \underset{x \doteq a}{L} \phi_n(x).$$
from Theorem III. Chapter I. But since each of the ϕ functions are, by hypothesis, continuous, we have.
$$\underset{x \doteq a}{L} \phi_1(x) = \phi_1(a); \underset{x \doteq a}{L} \phi_2(x) = \phi_2(a) \text{---} \underset{x \doteq a}{L} \phi_n(x) = \phi_n(a).$$
and equation (1) becomes
$$\underset{x \doteq a}{L} f(x) = \phi_1(a) + \phi_2(a) + \phi_3(a) + \text{------} + \phi_n(a).$$

But this right hand member of the equation is nothing but $f(a)$. Hence

$$\lim_{x \doteq a} f(x) = f(a),$$

and the function $f(x)$ is continuous. This theorem can not always be extended, however, to the case where n becomes infinite.

Theorem III: The product of a finite number of continuous functions is a continuous function.

Let

$$f(x) = \{\phi_1(x) \cdot \phi_2(x) \cdot \phi_3(x) \text{ - - - - - } \cdot \phi_n(x)\}$$

where $\phi_1(x)$, $\phi_2(x)$ - - - - $\phi_n(x)$ are each continuous functions. Then

$$\lim_{x \doteq x_1} f(x) = \lim_{x \doteq x_1} \{\phi_1(x) \cdot \phi_2(x) \text{ - - - } \phi_n(x)\}.$$

$$= \lim_{x \doteq x_1} \phi_1(x) \cdot \lim_{x \doteq x_1} \phi_2(x) \text{ - - - - - } \cdot \lim_{x \doteq x_1} \phi_n(x)$$

$$= \phi_1(x_1) \cdot \phi_2(x_1) \cdot \phi_3(x_1) \text{ - - - - } \cdot \phi_n(x_1)$$

(1) $$= f(x_1)$$

Since equation (1) is the condition for continuity, the theorem has been proved.

We may state the following as a corollary of the above theorem.

Cor. Any finite integral power of a continuous function is continuous.

The above theorem and corollary do not necessarily hold for an infinitely large value of n, where n is the number of terms or the power of the continuous function.

Theorem IV. *The quotient of two continuous functions in their common interval of continuity is continuous except at the values of the variable for which the denominator is zero.*

Let $f(x) = \frac{\phi(x)}{\psi(x)}$, where $\phi(x)$ and $\psi(x)$ are each continuous and where $\psi(x)$ differs from zero. Then

$$\lim_{x \doteq x_1} f(x) = \lim_{x \doteq x_1} \left\{ \frac{\phi(x)}{\psi(x)} \right\} = \frac{\lim_{x \doteq x_1} \phi(x)}{\lim_{x \doteq x_1} \psi(x)} = \frac{\phi(x_1)}{\psi(x_1)} = f(x_1).$$

The above condition holds because of Theorem V. Chapter I.

Theorem V. *If $f(x)$ is a continuous function then $a^{f(x)}$ is continuous.*

Since by Theorem XI. Chapter I

$$\lim_{x \doteq a} a^{f(x)} = a^{\lim_{x \doteq a} f(x)},$$

and because $f(x)$ is continuous, we have

$$a^{\lim_{x \doteq a} f(x)} = a^{f(a)}.$$

Then $a^{f(x)}$ is continuous when $f(x)$ is a continuous function. In this case a

is taken as a positive constant.

Theorem VI. *If $f(x)$ and $\phi(x)$ are continuous and if $f(x)$ is positive then if*

$$y = [f(x)]^{\phi(x)},$$

y is a continuous function.

If $y = [f(x)]^{\phi(x)}$ we may write

1) $$\log y = \phi(x) \log f(x).$$

2) $$\lim_{x \doteq a} \log y = \lim_{x \doteq a} [\phi(x) . \log f(x)]$$

$$\log \lim y = \lim_{x \doteq a} \phi(x) . \log \lim_{x \doteq a} f(x)$$

$$= \phi(a) . \log f(a).$$

from Theorem VIII. Page 21. Then since by Theorem III. Page 41, the product of two continuous functions is continuous we have proved that $\log y$ is a continuous function, where $y \neq 0$. Then since

$$\log \lim y = \phi a . \log f(a)$$

we may write

$$\lim y = [f(a)]^{\phi(a)}$$

and y has been shown to be continuous.

Theorem VII. *If $f(x)$ is continuous and different from zero, then $\frac{1}{f(x)}$ is also continuous.*

Since $f(x)$ is continuous it follows that

$$\lim_{x \doteq a} f(x) = f(a).$$

Then

$$\lim_{x \doteq a} \frac{1}{f(x)} = \frac{1}{\lim_{x \doteq a} f(x)} = \frac{1}{f(a)}.$$

then $\frac{1}{f(x)}$ is a continuous function.

Theorem VIII. *When a continuous function is defined for a set of points everywhere dense, then it is completely defined.*

In Theorem I. chapter I. it has been shown that if the limit

$$\lim_{x=a} f(x)$$

exists and equals some definite number A, then when we consider a set of points dense at $x=a$, we have

$$\lim_{x_n=a} f(x_n) = A.$$

We shall apply this theorem here. Because $f(x)$ is continuous we have

$$\lim_{x=a} f(x) = f(a).$$

Because of the theorem just mentioned we have

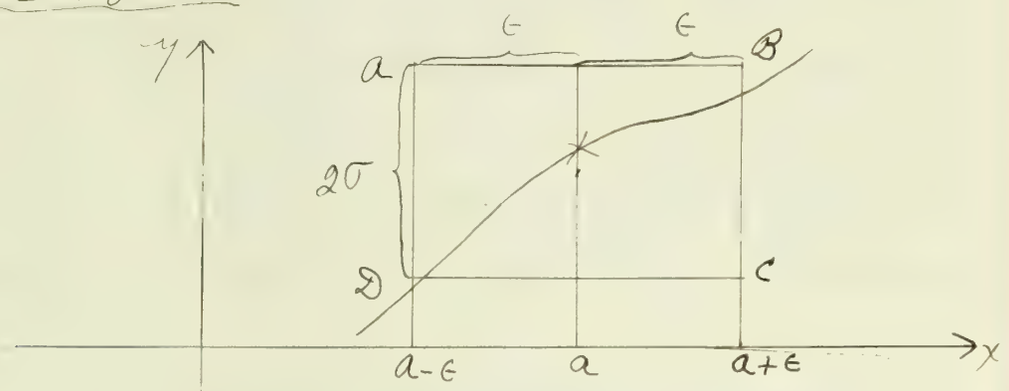
$$\lim_{x_n=a} f(x_n) = f(a).$$

That is to say, $f(x)$ is completely defined at $x=a$. If a be any point within the given interval of continuity of $f(x)$, that is to say, if $f(x)$ is known at a set of points everywhere dense in this interval, it follows that the continuous function $f(x)$ is completely defined for this interval.

Theorem IX. If the function $f(x)$ is continuous in the interval (α, β) then

$$\lim_{x \geq x_0} |\text{Oscillations}| = 0.$$

where $\alpha \leq x_0 \leq \beta$.



If we take the interval from a to $a+\epsilon$ or $a-\epsilon$, then we would have geometrically the above. Now if $D \equiv$ Oscillation of $f(x)$ in the interval (α, β), and if x_1 and x_2 are any two values of x in the interval we would have

$$D \geq |f(x_2) - f(x_1)|.$$

D cannot be greater than $2|f(x_2) - f(x_1)|$, because if it were greater, the function could not be contained within the rectangle A B C D whose dimensions are defined as follows. The value σ is chosen arbitrarily small. Then an ϵ can be so chosen that the value of the function for all values of x within the interval $(a-\epsilon, a+\epsilon)$ shall lie inside

the given rectangle, having the dimensions 2ϵ and 2σ. Therefore the limit of $f(x)$ could not exist in the neighborhood of the point a. But this is contrary to our hypothesis, for the function $f(x)$ is continuous throughout the interval, and hence the limit of $f(x)$ must exist. Now we have from this property of continuity

1.) $$\lim_{x_1 \doteq x_2} f(x_1) = f(x_2)$$

and

2.) $$\lim_{x_1 \doteq x_2} |D| = \lim_{x_1 \doteq x_2} |f(x_1) - f(x_2)|$$

But from (1)

$$\lim_{x_1 \doteq x_2} |f(x_1) - f(x_2)| = \left| \lim_{x_1 \doteq x_2} f(x_1) - \lim_{x_1 \doteq x_2} f(x_2) \right|$$
$$= |f(x_2) - f(x_2)| = 0,$$

Hence from (2)

$$\lim_{x_1 \doteq x_2} |D| = 0.$$

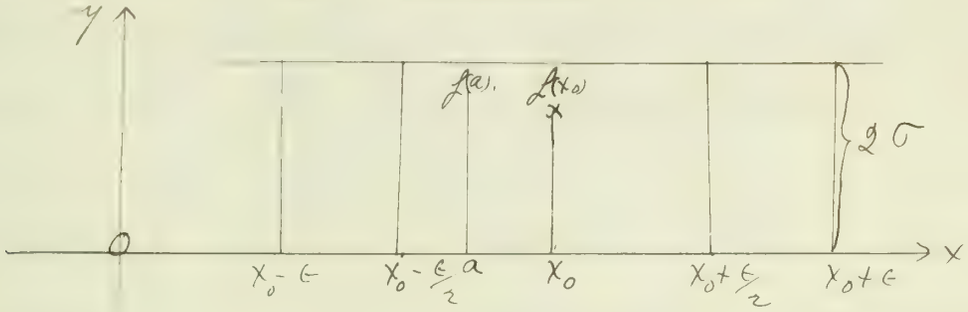
Since x_1 and x_2 were any two points of the interval (α, β), x assuming every value in the interval, and since the interval $(a-\epsilon, a+\epsilon)$ was any sub-interval of (α, β) we may write

$$\lim_{x \doteq x_2} |\text{Oscillation}| = 0$$

and the theorem is proved.

Theorem X. Every function which is

continuous throughout an interval, end points included, is uniformly continuous in the interval.



Select an arbitrarily small but constant value σ, and some point a. Then select an ϵ which is to be arbitrarily small. Since the function is continuous throughout the interval, we have for every point a the condition

$$\lim_{x \doteq a} f(x) = f(a).$$

Our proposition affirms that for an arbitrarily small but constant σ, $\epsilon(x)$ has a lower limit within the given interval, such that for every neighborhood of this point, however small, the lower limit of the ϵ's corresponding to the various points must be ϵ', where $\epsilon' \neq 0$. But by hypothesis the given function is continuous at x_0, and hence we have

$$\lim_{x \doteq x_0} f(x) = f(x_0),$$

for every point of the interval $(x_0 - \epsilon_0, x_0 + \epsilon_0)$
Consequently, within this interval the oscillation must be less than 2δ. If we take $x = a$ to be any point of the interval $(x_0 - \frac{\epsilon_0}{2}, x_0 + \frac{\epsilon_0}{2})$ we have

$$\lim_{x \doteq a} f(x) = f(a)$$

Now for every point within this interval $(x_0 - \frac{\epsilon_0}{2}, x_0 + \frac{\epsilon_0}{2})$ the corresponding value of $\epsilon(x)$ is equal to or greater than $\frac{\epsilon_0}{2}$, because, if this were not true, there would be an interval in which $f(x)$ would not be continuous. i.e. for which

$$\lim_{x \doteq a} f(x) \neq f(a).$$

which is contrary to our hypothesis. But we have seen that for the points within each interval including x_0, the function $\epsilon(x)$ has the lower limit ϵ', and from above it follows that

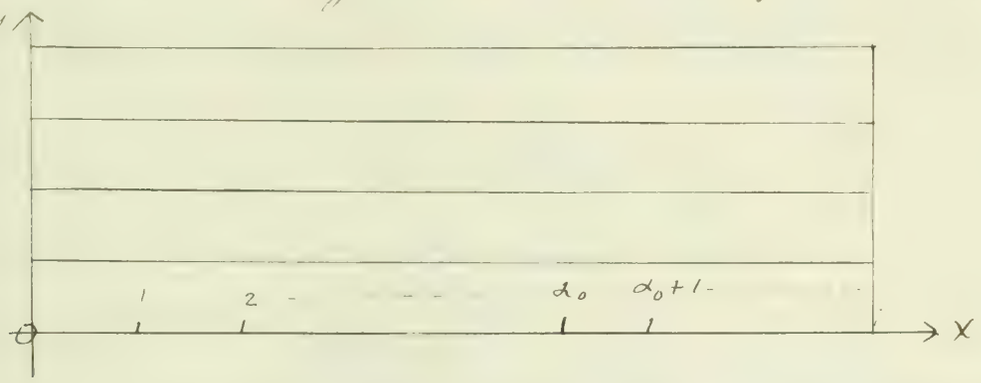
$$\epsilon' \geq \frac{\epsilon_0}{2} \neq 0,$$

because $\frac{\epsilon_0}{2}$ was some finite quantity different from zero. We may now replace the function $\epsilon(x)$ by $\frac{\epsilon_0}{2}$ for all values of x in the interval for which our function was defined as continuous and the function is uniformly continuous within the given interval.

Theorem XI. *When $f(x)$ is a finite, single valued, continuous function in an interval (α, β) there exists a point x_0, where $\alpha \leq x_0 < \beta$, such that $\lim_{x \doteq x_0} f(x)$ equals the upper limit of $f(x)$.*

Proof: We shall divide the proof into two parts.

I. To show that $f(x)$ has an upper limit.



From 0 lay off the unit segments 1, 2, 3 In some one of these intervals, besides values greater than $f(x)$ there must be present at least one value of $f(x)$. Call this interval $(\alpha+\alpha_0, \alpha+\alpha_0+1)$. Divide this interval into ten equal parts and in one of these there will be values greater than $f(x)$, but yet which contains at least one value of $f(x)$. Call this interval $(\alpha+\alpha_0+\frac{\alpha_1}{10}, \alpha+\alpha_0+\frac{\alpha_1}{10}+\frac{1}{10})$. Finally we have defined the following functions

$$\phi(x) = \alpha_0 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \frac{\alpha_3}{10^3} + \cdots\cdots + \frac{\alpha_n}{10^n}$$

$$\psi(x) = \alpha_0 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \frac{\alpha_3}{10^3} + \cdots\cdots + \frac{\alpha_n}{10^n} + \frac{1}{10^n}.$$

These are both regular sequences and have limits, as every such sequence does. These limits are equal because their corresponding terms differ by $\frac{1}{10^n}$, the limit of which approaches zero as n increases. Hence

1.) $$\underset{n=\infty}{L} \Phi(n) = \underset{n=\infty}{L} \Psi(n).$$

Let $\overline{W}\, L\, f(x)$ represent the upper limit of $f(x)$ i. e. provided it exists. Then we have

$$\Phi(n) \leq \overline{W}\, L\, f(x) \leq \Psi(n).$$

Since (1) is true and since

$$\underset{n=\infty}{L} \Phi(n) = \underset{n=\infty}{L} \Psi(n) = A,$$

where A is some definite integer, we have by Theorem V I. Chapter I.

$$\overline{W}\, L\, f(x) = A.$$

II. To show that $\underset{x=a}{L} f(x) = A$.

Let y be defined for all x's in the interval. Divide this interval (α, β) into the unit segments 1, 2, 3 - - - - - . In each of these subdivisions there is an upper limit, none of which may be greater than A, and for one interval, the upper limit is A. Call this interval $\alpha_\delta, \alpha_\delta + 1$. Subdivide this interval into 10 equal parts and in each of these sub inter-

vals there exists an upper limit which may not be greater than A and in one is equal to A. Call this α_1, $\alpha_0 + 1$. Proceeding in this way until we have formed n such sub-divisions we have defined the two functions,

$$\phi(n) = \alpha_0 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \frac{\alpha_3}{10^3} + \cdots\cdots + \frac{\alpha_n}{10^n}$$

$$\psi(n) = \alpha_0 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \frac{\alpha_3}{10^3} + \cdots\cdots + \frac{\alpha_n}{10^n} + \frac{1}{10^n}.$$

For some value of x, such that

$$\phi(n) \leq x \leq \psi(n),$$

the value of $f(x) = A$. Let

$$\lim_{n=\infty} \phi(n) \leq x_0 \leq \lim_{n=\infty} \psi(n) = A.$$

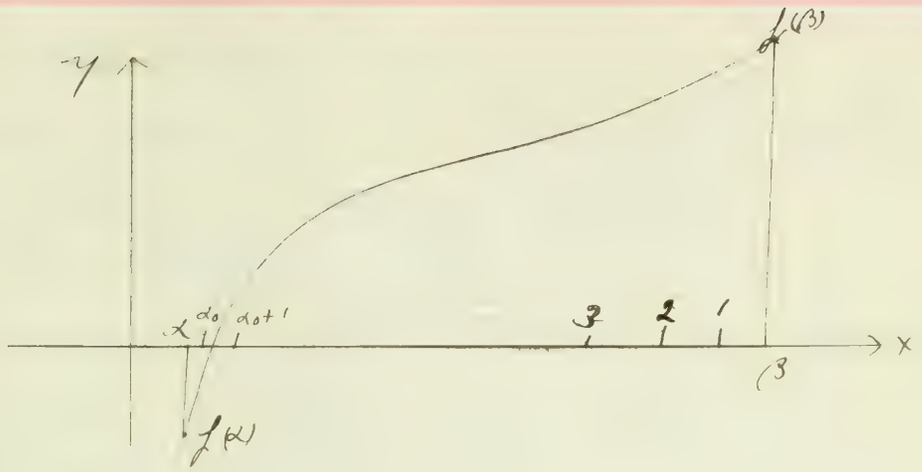
As x approaches x_0, $f(x)$ approaches A. or

$$\lim_{x=x_0} f(x) = A.,$$

and the theorem is completely demonstrated.

Theorem XII. If $f(x)$ is single valued and continuous in the interval (α, β), $\alpha \leq x \leq \beta$, and if the value of $f(\alpha)$ and $f(\beta)$ have opposite signs, then there is at least one value c, within the interval (α, β), for which $f(c) = 0$.

We may assume without loss of generality that $f(\alpha) < f(\beta)$. Let us lay off in the interval (α, β) unit distances.



In some one of these sub-intervals, there will exist values of the function, some of which are positive, while others are negative. Call this interval $\alpha + \alpha_0, \alpha + \alpha_0 + 1$. Now divide this sub-interval into 10 equal parts. In some one of these new sub-intervals there will be some values of the function which are positive and some which are negative. Call this interval $(\alpha_0 + \alpha_0 + \frac{\alpha_1}{10}, \alpha + \alpha_0 + \frac{\alpha_1}{10} + \frac{1}{10})$. Now divide this new interval into 10 equal parts and the same condition exists for some one of these parts that existed for the larger interval. If we repeat this process n times we may define the following functions.

$$\phi(n) = \alpha + \alpha_0 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \frac{\alpha_3}{10^3} + \cdots + \frac{\alpha_n}{10^n}$$

$$\psi(n) = \alpha + \alpha_0 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \frac{\alpha_3}{10^3} + \cdots + \frac{\alpha_n}{10^n} + \frac{1}{10^n}.$$

We now have the following condition existing

1) $\phi(n) < x < \psi(n)$,

where x is a point of the interval just defined. Then we may write

2) $f[\phi(n)] < f(x) < f[\psi(n)]$.

Since the above inequality holds the following condition is true. i. e.

3) $0 \leq f(x) \leq f[\psi(n)] - f[\phi(n)]$.

If we let n increase without limit, we shall have the following limits existing and having some definite value. i. e.

$$\lim_{n \doteq \infty} \phi(n) \quad \text{and} \quad \lim_{n \doteq \infty} \psi(n).$$

We know that these limits do exists, because both functions increase monotone within finite limits, and all such functions have limits. These limits are equal because the two functions differ in value only by $\frac{1}{10^n}$ and $\lim_{n \doteq \infty} \frac{1}{10^n} = 0$. Then we have

$$\lim_{n \doteq \infty} \phi(n) = \lim_{n \doteq \infty} \psi(n) = C$$

Hence when $n \doteq \infty$, $x \doteq C$. We have now

4) $\lim f(x) < \lim \{ f[\psi(n)] - f[\phi(n)] \}$

$= \lim f[\psi(n)] - \lim f[\phi(n)]$

$= 0$.

But since the function of x is continuous, we have

$$\lim_{x \doteq C} f(x) = f(C).$$

But since from (4)

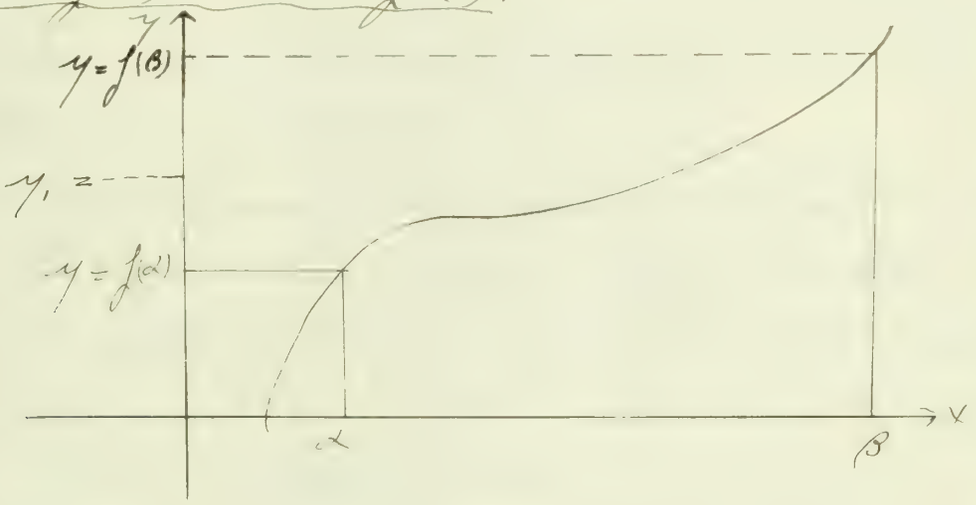
$$\lim_{x \doteq c} f(x) = 0$$

then

5). $$\lim_{x \doteq c} f(x) = f(c) = 0.$$

Since c was some point of the interval (α, β), we have shown that at at least one point, the value of the function becomes zero and the theorem has been proved.

Theorem XIII. When $y = f(x)$ is a single valued continuous function of x in the interval (α, β), α and β included, then as x passes through all values between α and β, $f(x)$ passes through all values between $f(\alpha)$ and $f(\beta)$.



Let y_1 be any value between $f(\alpha)$ and $f(\beta)$ and we will assume for convenience $f(\alpha) < f(\beta)$, and both positive. Lay off on the x-axis, in the interval (α, β) unit distances. As x increases from α toward β,

we reach somewhere an interval where not all of the corresponding functions are less than y_1. Call this interval $(\alpha + \alpha_0, \alpha + \alpha_0 + 1)$. Divide this interval into 10 equal parts. As x increases within the interval $(\alpha + \alpha_0, \alpha + \alpha_0 + 1)$ we shall again reach one of the new intervals, where not all of the values of the function are less than y_1. Let this new interval be $\left(\alpha + \alpha_0 + \frac{\alpha_1}{10}, \alpha + \alpha_0 + \frac{\alpha_1}{10} + \frac{1}{10}\right)$. If we proceed in this same way forming n divisions, we shall define the following functions.

$$\Phi(n) = \alpha + \alpha_0 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \frac{\alpha_3}{10^3} + \cdots + \frac{\alpha_n}{10^n}$$

$$\Psi(n) = \alpha + \alpha_0 + \frac{\alpha_1}{10} + \frac{\alpha_2}{10^2} + \frac{\alpha_3}{10^3} + \cdots + \frac{\alpha_n}{10^n} + \frac{1}{10^n}.$$

where these functions are such that

1.) $$\Phi(n) \leq x \leq \Psi(n).$$

For some x within this interval (Φ, Ψ), $f(x)$ has by Theorem XI, Page 49, an upper limit which we will call $W(x)$ and a lower limit $L(x)$ so related that

2.) $$L(x) \leq y_1 \leq W(x).$$

Since both of the functions $\Phi(n)$ and $\Psi(n)$ are monotone, their limits exist as n increases indefinitely. Since the functions differ only by an amount equal to $\frac{1}{10^n}$ their limits are equal as

$$\lim_{n \doteq \infty} \frac{1}{10^n} = 0.$$

Then we have

3.) $$\lim_{n \doteq \infty} \phi(n) = \lim_{n \doteq \infty} \psi(n)$$

From (1) we then have

4.) $$\lim_{n \doteq \infty} \phi(n) = \lim_{n \doteq \infty} \psi(n) = x_0,$$

where x_0 is a definite value of x such that $\alpha \leq x_0 \leq \beta$. Moreover it follows from Theorem IX, Page 45, that

5.) $$\lim_{X \doteq X_0} (W(X) - L(X)) = 0.$$

Then

6.) $$\lim_{X \doteq X_0} W(X) = \lim_{X \doteq X_0} L(X) = \lim_{X \doteq X_0} f(X),$$

by Theorem VI. chapter I. Since $f(x)$ is continuous in the interval (α, β), we have

7.) $$\lim_{X \doteq X_0} L(X) = f(x_0).$$

Then from (5) and (6) it follows that

8.) $$y_1 = f(x_0).$$

and the theorem is proved since x_0 is a definite point of the interval (α, β) and y_1 is any value of $f(x)$ between $f(\alpha)$ and $f(\beta)$.

The following corollary follows at once from the above Theorem.

Corollary: If $f(x)$ is continuous between the points α and β, then it takes, at least once, every value between the Maximum and Minimum values of the function within this interval.

Theorem XIV. *If $y = f(x)$ is single valued and continuous within the interval (α, β), the end points included, and if x increases monotone, then the inverse function, $x = \varphi(y)$, is a single valued, continuous function and increases monotone with y.*

By the previous theorem we know that as $x \doteq x_0$, y approaches simultaneously a value y_0 and conversely. Then it follows that

1) $$\lim_{x \doteq x_0} x = x_0 = \lim_{y \doteq y_0} \varphi(y).$$

Since $f(x)$ is continuous within the interval (α, β), we have

2) $$\lim_{x \doteq x_0} f(x) = f(x_0) = y_0$$

and

3) $$x_0 = \varphi(y_0).$$

Then it follows that

4) $$\lim_{y \doteq y_0} \varphi(y) = x_0 = \varphi(y_0)$$

from (1) and (3). Since the above conditions hold, we have from equation (4) the condition for the continuity of $\varphi(y)$.

If we select any two values of x within the given interval (α, β), for instance $x' \neq x''$, then

$$y' = f(x'), \quad y'' = f(x''), \quad y' \neq y''.$$

Since x increases monotone, it follows that $y = f(x)$ has only one solution, i. e. $x = \varphi(y)$ is a single valued function. That this solution exists follows directly from the preceding theorem. By our hypothesis, y increases monotone as x increases. Therefore it follows from the fact that $x = \varphi(y)$ is a single valued function, that $\varphi(y)$ increases monotone with y. Then is the theorem demonstrated.

Theorem XV. If we have $y = f(u)$ and $u = \varphi(x)$, when $f(u)$ and $\varphi(x)$ are continuous functions, then y is a continuous function of x.

Let

1.) $$y = f(u) = f[\varphi(x)] = \psi(x)$$

By Theorem XIII Page 53-4, we know that as x approaches x_0, u approaches u_0 simultaneously. Then, since $\varphi(x)$ and $f(u)$ are continuous functions, it follows that

2.) $$\lim_{x \doteq x_0} \varphi(x) = \varphi(x_0)$$

and

3.) $$\lim_{u \doteq u_0} f(u) = f(u_0)$$

4.) $$\lim_{x \doteq x_0} \varphi(x) = \varphi(x_0) = \lim_{u \doteq u_0} u = u_0.$$

Then

$$u_0 = \varphi(x_0)$$

$$y_0 = f(u_0) = \psi(x_0),$$

and it follows that

$$4) \quad \lim_{x \doteq x_0} \psi(x) = \lim_{x \doteq x_0} f[\varphi(x)] = \lim_{u \doteq u_0} f(u) = f(u_0) = y_0.$$

But

$$y_0 = \psi(x_0)$$

and

$$\lim_{x \doteq x_0} \psi(x) = \psi(x_0).$$

This last is the condition for continuity and we have shown that y is a continuous function of x.

We may state in connection with this theorem the following

Ex. 1. If $f(x)$ is a continuous function then $\log f(x)$ is continuous at the point $x = a$, when $f(a) \neq 0$.

For

$$\lim_{x \doteq a} \log f(x) = \log \lim_{x \doteq a} f(x)$$

$f(x)$ being positive and different from 0. Then since $f(x)$ is continuous by hypothesis

$$\lim_{x \doteq a} f(x) = f(a) \neq 0$$

and we have

$$\log \lim_{x \doteq a} f(x) = \log f(a).$$

Then

$$\lim_{x \doteq a} \log f(x) = \log f(a)$$

and the example is shown to be true.

Theorem XVI. When $f(x)$ is continuous in the interval (α, β), and points included

and takes in this interval values which are numerically less than an arbitrarily small number, then the function takes, for some definite value of x in the same interval the value zero.

Since the function is continuous we have the following condition.

1.) $$\lim_{X = X_n} f(X) = f(X_n).$$

By hypothesis, $f(X) < \sigma$, where σ is some arbitrarily chosen number which may be made as small as we please. Now let us select a set of values $\sigma_1, \sigma_2, \sigma_3 \cdots \sigma_n$ such that

2) $$\lim_{n = \infty} \sigma_n = 0$$

Since we have an infinite number of values $x_1, x_2, x_3 - - - - -$, they must become dense at some point. Let this point be x_0. Because the limit $\lim_{X = X_0} f(X)$ exists, we must obtain the same limiting value of $f(X)$ when $x = x_0$ through the values $x_1, x_2, x_3 \cdots x_n$. We have then

$$\lim_{n = \infty} f(x_n) \leq \lim_{n = n} \sigma_n$$

Since $x_n = x_0$ as $n = \infty$, we may write this

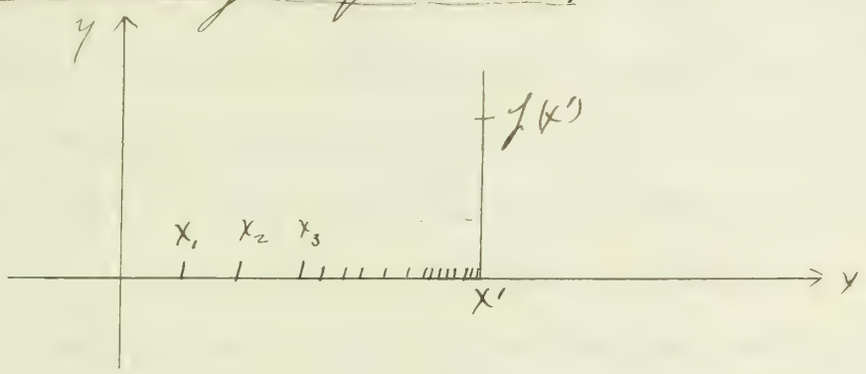
$$\lim_{x_n = x_0} f(x_n) \leq \lim_{n = \infty} \sigma_n$$

or

$$f(x_0) = 0.$$

The point x_0 must be a point of the interval (α, β) because that interval is by definition a closed interval and must include all limiting points.

Theorem XVII. Let the function $f(x)$ be continuous at the point x'. Also assume that there are a set of points whose distance from x' is less than any assignable quantity. If at each of these points, the function takes the value A or values which differ from A by less than some arbitrary number, then $f(x')$ always equals A.



Take a set of points $x_1\ x_2\ x_3 - - - - x_n - - - -$ dense at x', such that at each of these points the function $f(x)$ differs from A by less than some arbitrary number. Take this arbitrary number in general as η_n. We then have

$$f(x_n) = A \pm \eta_n.$$

But $f(x)$ is continuous at x' and hence

$$\lim_{x \doteq x'} f(x) = f(x').$$

From this it follows that we should obtain

the same limiting value if we let $x \doteq x'$ through any set of values dense at x', say the set $x_1, x_2, x_3 - - - - - x_n$. We have therefore in the limit, since $x_n \doteq x'$, as $n \doteq \infty$

$$\lim_{x_n \doteq x'} f(x_n) = \lim_{n \doteq \infty} (A \pm \sigma_n)$$

or

$$f(x') = A.$$

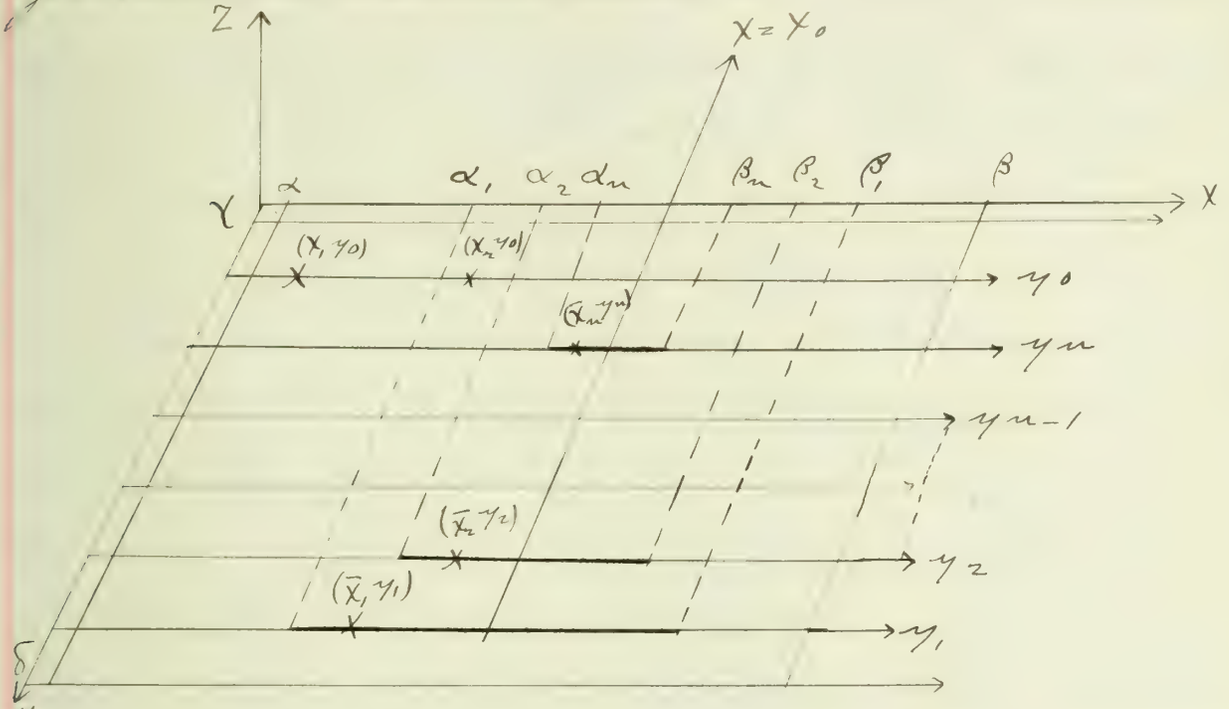
We have thus proved the theorem to be true.

Theorem XVIII The set of points at which the $(x\,y)$ discontinuity equals or exceeds an arbitrary number A, cannot be everywhere dense in any sub-region, providing the given function is continuous in each variable separately.

We shall prove this theorem by assuming that there are such points everywhere dense in some sub-region and show that this leads to a contradiction of the hypothesis that $f(x\,y)$ is continuous in each variable separately.

Let $D =$ a amount of $(x\,y)$ discontinuity. In this case we consider the point where $D \geqq A$. Now let the points at which the above condition holds be everywhere dense in the region $(\alpha\ \beta\ \chi, \delta)$. Then every point in this sub-region is such a point,

because when $f(xy)$ is continuous in each variable alone, then the points at which it has an (xy) discontinuity equal to or greater than an arbitrary number A, form a closed set.



Let $y = y_0$ be any line lying in part within this region and parallel to the x-axis. Every point upon this line and within this region is then a point at which the (x,y)-discontinuity is equal to or greater than A. Let (x_1, y_0) denote any point on $y = y_0$, where $\alpha < x_1 < \beta$. Since the discontinuity at (x_1, y_0) in the two variables $\geq A$, then the oscillations within any region about (x_1, y_0) must be greater than A. We can then surely find some point $(\bar{x}_1, y_1)$

lying in the neighborhood of (x_1, y_0) such that

$$| f(x_1, y_0) - f(\bar{x}_1, y_1) | \geq \frac{A}{3}.$$

Draw the line $y = y_1$. By hypothesis, $f(x_0, y_1)$ is a continuous function in x. Hence it is possible to find upon $y = y_1$, an interval (α_1, β_1) lying entirely within (α, β), such that for every x within it, we have

1.) $$\lim_{x = \bar{x}_1} f(x, y_1) = f(\bar{x}_1, y_1)$$

By the proper selection of this interval (α_1, β_1), it is possible to have for every x within it the property that

2.) $$| f(x_1, y_0) - f(x, y_0) | > \frac{A}{5}.$$

It is to be observed that α_1, β_1 are so taken that $\alpha < \alpha_1$, $\beta > \beta_1$. Within (α_1, β_1) select on $y = y_0$ any point (x_2, y_0). This again is a point at which the (x, y)-discontinuity $\geq A$. We may then find in the neighborhood of (x_2, y_0) a point $(\bar{x}_2, y_2)$ such that we have

3.) $$| f(x_2, y_0) - f(\bar{x}_2, y_2) | \geq \frac{A}{3}.$$

The point (x_2, y_2) must be found arbitrarily near to the point (x_2, y_0). It is sufficient for our purpose if it be so taken that y_2 lies between y_0 and y_1. Draw the line $y = y_2$. Upon this line there must exist an interval (α_2, β_2), lying within (α_1, β_1),

such that for every x within it we have

4.) $$\lim_{x \doteq \bar{x}_2} f(x, y_2) = f(\bar{x}_2, y_2).$$

As before, by the proper selection of this interval (α_2, β_2) we have the condition that for every point of the interval

5.) $$| f(x_2, y_0) - f(x_1, y_2) | > \tfrac{A}{5},$$

These same points lie within (α_1, β_1). Select again a point (x_3, y_0) lying within (α_2, β_2) and proceed as before. Continuing in this way, we have in general

6.) $$| f(x_n, y_0) - f(x_1, y_n) | > \tfrac{A}{5},$$

for $n = 1, 2, 3, \ldots\ldots$, and where $\alpha_n < x < \beta_n$ and y_n lies between y_0 and y_{n-1}. We have selected the α's and β's so that

$$\alpha < \alpha_1 < \alpha_2 < \alpha_3 \ldots\ldots < \alpha_n < \ldots\ldots$$

$$\beta > \beta_1 > \beta_2 > \beta_3 \ldots\ldots > \beta_n > \ldots\ldots$$

We now impose upon these numbers the farther condition, which we can do without loss of generality, that

$$\lim_{n \doteq \infty} \alpha_n = \lim_{n \doteq \infty} \beta_n = x_0.$$

x_0 then belongs to each of the intervals (α_n, β_n) and hence the points $x_1, x_2, x_3 \ldots x_n$ are dense at x_0. Because of the continuity of $f(x_1, y_0)$ we have then

7.) $$\lim_{x_n \doteq x_0} f(x_n, y_0) = f(x_0, y_0).$$

Since x_0 lies within every interval (α_n, β_n), $n = 1, 2, 3 \text{ ---- }$, we may write for inequality (6) the following

8.) $$| f(x_n, y_0) - f(x_0, y_n) | > \frac{A}{5}.$$

Since y_n approaches y_0 as x_n approaches x_0, we may write the following inequality.

9.) $$\Big| \lim_{x_n = x_0} f(x_n, y_0) - \lim_{y_n = y_0} f(x_0, y_n) \Big| > \frac{A}{5}.$$

But from (7)

$$\lim_{x_n = x_0} f(x_n, y_0) = f(x_0, y_0)$$

and we have from (9)

10) $$\lim_{y_n = y_0} f(x_0, y_n) \neq f(x_0, y_0),$$

for all values of $f(x_0, y_n)$ must differ from $f(x_0, y_0)$ by more than $\frac{A}{5}$ as n increases without limit. From (10) we see that $f(x,y)$ defined by our hypothesis can not be continuous in y at the point (x_0, y_0). Hence, from this contradiction of the hypothesis, the theorem is proved.

Theorem XIX. <u>When $f(x,y)$ is continuous with respect to each variable separately the points at which it is continuous with respect to both form a set everywhere dense.</u>

Let $(\alpha_0\ \beta_0\ \gamma_0\ \delta_0)$ be any sub region of the given region for which $f(x, y)$ is defined.

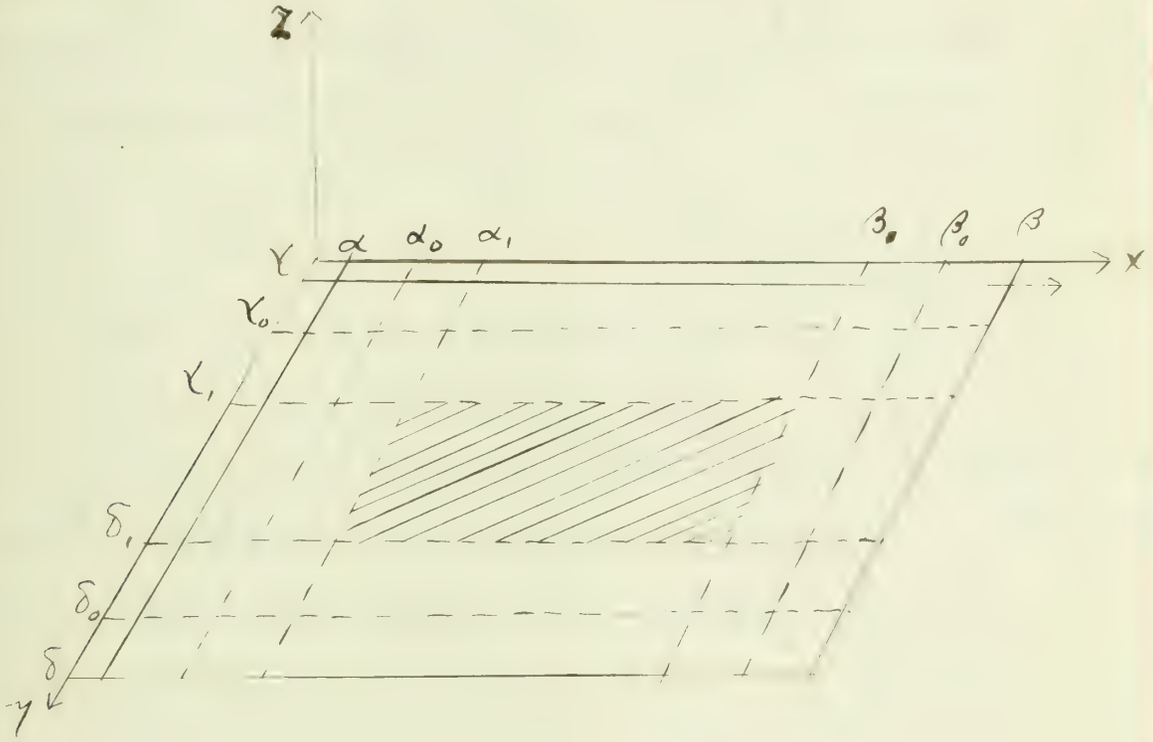
Consider any sequence

$$\sigma_1, \sigma_2, \sigma_3 \text{ - - - - - - } \sigma_n \text{ - - - - -}$$

having the property that

$$\lim_{n \doteq \infty} \sigma_n = 0.$$

It follows from the previous theorem that the points at which $f(x, y)$ has an (x, y) discontinuity greater than σ_1 cannot be everywhere dense in the sub-region $(\alpha_0, \beta_0, \gamma_0, \delta_0)$.



Hence there must be some portion of this region which is free from such points of discontinuity. Let $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ be such a sub-region, lying wholly within the previous one. The points at which $f(x, y)$ has an (x, y)-discontinuity greater than σ_2 cannot be everywhere dense within the region

$(\alpha_1, \beta_1, \gamma, \delta_1)$. There must exist, then, some subdivision of it, say $(\alpha_2\ \beta_2\ \gamma_2\ \delta_2)$ which is entirely free from such points. Now again the points at which $f(x, y)$ may have an (x, y) discontinuity greater than σ_3 cannot be everywhere dense in this interval etc. In general there must exist a sub-region $(\alpha_n\ \beta_n, \gamma_n, \delta_n)$ entirely free from any (x,y) discontinuities greater than σ_n. Moreover we have defined the sequences so that

$$\begin{aligned}
&\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \qquad\qquad \leq \alpha_n \leq \cdots \\
&\beta_0 \geq \beta_1 \geq \beta_2 \geq \beta_3 \cdots \cdots \geq \beta_n \geq \cdots \\
&\gamma_0 \geq \gamma_1 \geq \gamma_2 \geq \gamma_3 \cdots \cdots \geq \gamma_n \geq \cdots \\
&\delta_0 \leq \delta_1 \leq \delta_2 \leq \delta_3 \cdots \cdots \leq \delta_n \leq \cdots
\end{aligned}$$

These are all monotone and hence have limits. Let us assume further that

$$\lim_{n=\infty} \alpha_n = \lim_{n=\infty} \beta_n = x_0$$

$$\lim_{n=\infty} \gamma_n = \lim_{n=\infty} \delta_n = y_0.$$

It follows then that $f(x, y)$ has at the point (x_0, y_0) a discontinuity with respect to both variables equal to

$$\lim_{n=\infty} \sigma_n = 0$$

That is to say, the point (x_0, y_0) is a point of continuity with respect to both variables, equal to

$$\lim_{n=\infty} \sigma_n = 0.$$

That is to say, the point (x_0, y_0) is a point of continuity with respect to both variables As the sub-region $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ was any sub-region and this contained at least one point of continuity with respect to both variables, it then follows that these points of continuity must be at least every where dense as the theorem demands.

Theorem XX. In a point wise discontinuous function, if the points at which the function is constant form a set every where dense, then the points at which the function is continuous belong to this same set.

Let us denote by $M(m)$ the set of points where the function has a constant value c. Take any point x_1 at random such that

$$x = x_1 = \ldots$$

This point x_1 is either

1.) A point of $M(m)$ or

2.) It is not such a point.

Let us take the second case into consideration Then $f(x_1)$ will be a value of $f(x)$. The $M(m)$ points are every where dense in the interval

(α, β); therefore we can talk about the limit

$$\lim_{m \doteq x_1} f(m).$$

Moreover, for the same reason, we can approach x_1, through any set dense at x_1. Then

$$\lim_{m \doteq x_1} f(m) = \lim_{m \doteq x_1} c = c \neq f(x_1),$$

for, by hypothesis, x_1 is not a point of $M(m)$, and consequently $f(x)$ cannot have at this point the constant value c. In this case then, x_1 is not a point of continuity. Since x_1 represents any point, no such point can belong to $M(m)$ if it is a point of discontinuity. On the other hand, all other points must belong to the set $M(m)$ because in this case

$$\lim_{m \doteq x_1} f(m) = \lim_{m \doteq x_1} c = c = f(x_1)$$

Therefore, since the points of continuity are the only other points in the interval, they must necessarily belong to the set $M(m)$. Hence the theorem.

Theorem XXI. *If the points of discontinuity greater than an arbitrary number A form a set which is in no sub-interval everywhere dense, the function is point wise discontinuous.*

Let us take any set of decreasing numbers,

$$a_0 > a_1 > a_2 > a_3 - - - - - > a_n > \quad .$$

such that

$$\lim_{n \doteq \infty} a_n = 0.$$

as the successive values of n. Let M_n be the assemblage of points where the function has a discontinuity greater than or equal to a_n. This gives rise to a series of assemblages

$$M_0 \; M_1 \; M_2 - - - - - - - M_n - - - - -,$$

each containing as sub-assemblages all of those preceding it. As $a_n \doteq 0$, these assemblages absorb more and more all of the points of discontinuity of the given function. It can happen that as $n \doteq \infty$, the assemblage M_n will penetrate every sub interval and become everywhere dense in the interval of definition. However, this does not need to be the case. In any case, it is possible to show that the points of continuity lie everywhere dense. We do this as follows. Let (α, β) be any interval. Then either the entire interval (α, β) is free from points of M_0 or some

sub-interval (α_0, β_0) is, by Theorem XVIII, Page 62. For the same reason, there is some sub-interval of (α_0, β_0), say (α_1, β_1), which is entirely free from points of the ensemble M_1. Continuing in this way, we obtain the two sequences

$$\alpha_0, \alpha_1, \alpha_2 \quad \cdots \quad \alpha_n \quad \cdots\cdots$$
$$\beta_0, \beta_1, \beta_2 \quad \cdots \quad \beta_n \quad \cdots\cdots$$

which define the end points of these successive sub-intervals. These are each regular sequences and hence define two numbers α' and β' respectively. These numbers which are the limits of the two sequences may be equal or they may be different. If we have

$$\lim_{n=\infty} (\alpha) = \alpha', \quad \lim_{n=\infty} (\beta) = \beta'$$
$$\alpha' = \beta' = x_0,$$

then x_0 is a point of continuity, in the second case, i. e.

$$\alpha' \neq \beta',$$

we have an interval $(\alpha' \beta')$ which contains none but points of continuity. In any case, the interval (α, β) contains at least one point of continuity. As (α, β) was <u>any</u> interval within the definition of the function, it follows

that the points of continuity are everywhere dense. This property is the characteristic feature of a point wise discontinuous function, and with this our proposition is established.

Theorem XXII. *If $f(x)$ has a discontinuity equal to or greater than A, at a set of points dense at $x = x_0$, then it has such a discontinuity at $x = x_0$ itself.*

Let $x_1\ x_2, x_3\ x_4 \text{ - - - - } x_n \text{ - - -}$ be a set of points dense at $x = x_0$, at each point of which the discontinuity is equal to or greater than A, where A is some arbitrary number. Within an arbitrarily small neighborhood of each x_n, we have values of the function which differ by an amount $\geq A$. Of these values select $\xi_1, \xi_2 \text{ - - - } \xi_n$ and $\bar{\xi}_1, \bar{\xi}_2 \text{ - - - } \bar{\xi}_n$ such that $f(\xi_1), f(\xi_2) \text{ - - - -}$ and $f(\bar{\xi}_1), f(\bar{\xi}_2) \text{ - - - -}$ form regular sequences. This is possible, because we can find in the neighborhood of $x_1, x_2 \text{ - - -}$ some point such that each of the values $f(\bar{\xi}_1), f(\bar{\xi}_2) \text{ - - - -}$ and $f(\xi_1), f(\xi_2) \text{ - - - - - -}$ is less than some finite number G. and such that from some point on, the following condition will

exist. i. e.

$$| f(\xi_{n+\nu}) - f(\xi_\nu) | < \epsilon \quad , \quad n = 1, 2, 3$$

Then since the two properties just given are those of regular sequences, we have shown that we have formed two regular sequences i. e.

$$f(\xi_1), \; f(\xi_2) \; f(\xi_3) - - - - - - - - f(\xi_m)$$
$$f(\bar{\xi}_1), \; f(\bar{\xi}_2), \; f(\bar{\xi}_3) - - - - - - - - f(\bar{\xi}_m) \; .$$

Then since every regular sequence has a limit we have

$$\lim_{\xi_m = x_0} f(\xi_m) = C$$

$$\lim_{\bar{\xi}_m = x_0} f(\bar{\xi}_m) = B,$$

where C and B are each some finite number. Since the set of points $x_1, x_2 \cdots x_m$ are dense at x_0 then the two sets $\xi_1, \xi_2 \cdots \xi_m$ and $\bar{\xi}_1, \bar{\xi}_2, \cdots \bar{\xi}_m$ are also dense at x_0. But since $x_1 x_2 \cdots x_m$ above

$$D(\xi) \geq A,$$

where $D(\xi)$ represents the amount of discontinuity, then we have

$$| f(\xi_m) - f(\bar{\xi}_m) | \geq A$$

and

$$| \lim_{\xi_m = x_0} f(\xi_m) - \lim_{\bar{\xi}_m = 0} f(\bar{\xi}_m) | \geq A.$$

Then evidently

$$\frac{1}{\xi_n \neq x_0} \lambda(\xi_n) \neq \frac{1}{\xi_n = x_0} \lambda(\xi_n),$$

and at x_0 the amount of discontinuity is equal to or greater than δ.

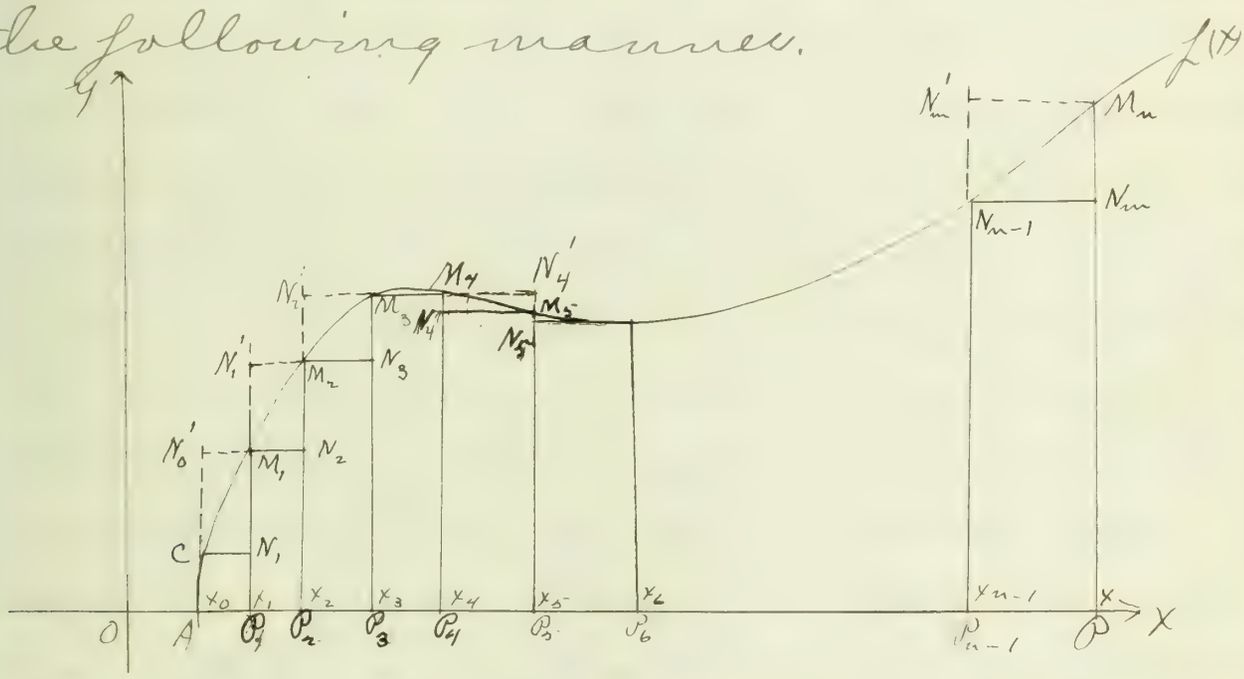
The following corollary follows at once from the above theorem.

Corollary. In a point-wise discontinuous function the points where the function has a discontinuity greater than an arbitrary number δ, form a closed set which is in no sub-interval everywhere dense.

§ 3. The Definite Integral and Some of its Properties.

The purpose of this section is to base the definition and the properties of the definite integral upon limits directly. We shall first define a definite integral upon this basis.

If we have the continuous function $f(x)$ and wish to consider the area bounded by the curve, the axis of x and the ordinates at A and P, we proceed in the following manner.



Let us break up the projection of the curve on the x-axis into n intervals and multiply each of these intervals by the ordinate at its beginning. If we let $E_n(x, x_0)$ represent

this product, we now have.

$$1)\ F_n(X, X_0) = f(X_0)(X_1 - X_0) + f(X_1)(X_2 - X_1) + f(X_2)(X_3 - X_2) + \text{+ - - - - - - + } f(X_{n-1})(X - X_{n-1}).$$

If as is the case with the polygons $A, C, N, P_1, P_1, M_1, N_2, P_2, \quad P_3, N_3, M_2 P_3 \quad \text{-} P_{n-1} N_{n-1} M_n P,$ the ordinate at the beginning of each interval is the minimum value of the function in the interval, then the sum of these products, i.e $f(X_0)(X_1 - X_0)$ etc. gives the value of the inscribed polygons. But if the first ordinate is the maximum value of all those in the interval, then the sum of the products, referred to above, would give the value of the area of the circumscribed polygons. This is the case with the polygon $P_4 M_4 N_4' P_5$ in our figure. In case the function is such that we have both these conditions existing as we have in the figure given here, then the value of $F_n(X, X_0)$ must necessarily not be larger than the circumscribed area, nor smaller than the inscribed area. We may state the same thing as follows. Let us take in the interval (X_0, X), g_0 as the minimum and G_0 as the maximum value of $f(X)$; in the interval

(x_2, x_1) g_1 as the minimum and G_1, the maximum value of $f(x)$; and, last of all, in the interval (x, x_{n-1}), let g_{n-1} and G_{n-1}, be the minimum and maximum values, respectively, of $f(x)$. Then, if as before

$$\overline{f}_n(x, x_0) = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \cdots + f(x_{n-1})(x - x_{n-1}),$$

and, if we form the two functions $\phi_n(x_1, x_0)$ and $\psi_n(x_1, x_0)$ such that

$$(2)\quad \begin{cases} \phi_n(x_0, x) = g_0(x_1 - x_0) + g_1(x_2 - x_1) + \cdots + g_{n-1}(x - x_{n-1}) \\ \psi_n(x_0, x) = G_0(x_1 - x_0) + G_1(x_2 - x_1) + \cdots + G_{n-1}(x - x_{n-1}), \end{cases}$$

we may write

$$3.)\qquad \phi_n(x_0, x) \leq \overline{f}_n(x, x_0) \leq \psi_n(x_0, x)$$

If we now form the differences between the area of the circumscribed and inscribed polygons, we have

$$4.)\quad \psi_n(x_0, x) - \phi_n(x_0, x) = (G_0 - g_0)(x_1 - x_0) + (G_1 - g_1)(x_2 - x_1) + \cdots + (G_{n-1} - g_{n-1})(x - x_{n-1}).$$

The differences $(G_0 - g_0)$, $(G_1 - g_1)$, $(G_2 - g_2)$ $\cdots$ $(G_{n-1} - g_{n-1})$ we call the oscillation of the function. Now as n increases indefinitely, the limit of these differences approaches zero. If we have a set of values, $\sigma_0, \sigma_1, \sigma_2 \cdots \sigma_{n-1}$, such that $(G_0 - g_0) = \sigma_0$, $(G_1 - g_1) = \sigma_1$, $(G_2 - g_2) = \sigma_2 \cdots (G_{n-1} - g_{n-1}) = \sigma_{n-1}$; and such that the value σ is equal

to a greater than the upper limit of the values $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \cdots \cdots \sigma_{n-1}$. Then we may write (4) as follows.

5.) $\psi_n(x, x_0) - \phi_n(x, x_0) \leq (x - x_0)\,\sigma.$

Since $f(x)$ was taken as continuous, the limit of the oscillation is equal to zero. Then as n increases indefinitely, we know that σ approaches zero. Since $\psi_n(x, x_0)$ and $\phi_n(x, x_0)$ are each monotone functions each has a limit. Now we may write

6.) $$\lim_{n=\infty} \{\psi_n(x, x_0) - \phi_n(x, x_0)\} \leq \lim_{\sigma=0} (x - x_0)\,\sigma$$
$$= (x - x_0) \lim_{\sigma=0} \sigma$$
$$= 0.$$

Since

$$\lim_{n=\infty} \{\psi_n(x, x_0) - \phi_n(x, x_0)\} = 0$$

and the two limits $\lim_{n=\infty} \phi_n(x_0, x)$, and $\lim_{n=\infty} \psi(x_0, x)$ exist, we have

7.) $$\lim_{n=\infty} \psi_n(x_0, x) = \lim_{n=\infty} \phi_n(x_0, x).$$

It now remains to show that the results obtained above do not depend upon the manner of division of the interval. Let us divide any or all the intervals (x_0, x), (x_2, x_1) $\cdots$ (x, x_{n-1}) in any way we please. Then the smallest value of $f(x)$ in none of these smaller intervals

is smaller than the smallest value of $f(x)$ in the old interval. If we let $\phi(x_0; x_1, x_2 - - - - x_{n-1}; - -; x)$ represent the function formed, for this new division of the intervals, in the same way that $\phi_n(x_0, x)$ was formed in (2) Page 78. we have

8). $\phi(x_0; x_1, x_2 - - - x_{n-1} - - - ; x) \geq \phi(x_0; x_1, x_2 - - x_{n-1}; x)$

On the other hand, the largest value of $f(x)$ in none of the new sub-intervals can be larger than the largest value of $f(x)$ in the old interval. Then if we use a similar notation to that above we would always have

9.) $\psi(x_0; x_1, x_2 - - - - x_{n-1} ; x) \leq \psi(x_0; x_1 x_2 \cdot x_{n-1}; x)$.

The value of ϕ is then always equal to or larger than every other ϕ, and every ψ is always equal to or smaller than every other ψ. But yet, from the definition of these two functions, we always have every ψ larger than any ϕ. viz.

$\phi(x_0; x_1 x_2 - - - - x_{n-1}; x) \leq \psi(x_0; x_1', x_2' - - x_{n-1}'; x)$.

Since ψ is always decreasing and since ϕ is always increasing, they are both monotone functions and hence have limits. We may write

$$(10)\quad \begin{cases} \lim_{n\doteq\infty} \phi_n(X, X_0) = \phi(X, X_0) \\ \lim_{n\doteq\infty} \psi_n(X, X_0) = \psi(X, X_0) \end{cases}$$

Since we have from (7)

$$\lim_{n\doteq\infty} \psi_n(X_0, X) = \lim_{n\doteq\infty} \phi_n(X_0, X)$$

we have from (10) that

11.) $$\phi(X, X_0) = \psi(X, X_0).$$

If we indicate this common value by $F(X, X_0)$ we have, according to (3) Page 78

12.) $$\lim_{n\doteq\infty} F_n(X, X_0) = F(X, X_0)$$

Because if

$$\lim_{n\doteq\infty} \phi_n(X_0, X) = \lim_{n\doteq\infty} \psi_n(X_0, X) = F(X_0, X).$$

then, according to Theorem VI. Page 20, the limit of $F_n(X_0, X)$ as $n \doteq \infty$ would exist and equal $F(X_0, X)$.

We may state the results of the above demonstration in the following theorem.

Theorem I. *If $f(x)$ is continuous and positive in the interval (X_0, X) then the sum*

$$F_n(X, X_0) = f(X_0)(X_1 - X_0) + \text{- - - - - -} + f(X_{n-1})(X - X_{n-1})$$

in which

$$X_0 < X_1 < X_2 \text{- - - - - - - -} < X_{n-1} < X \text{- - - - - - - -}$$

possesses a limit independent of the manner of division of the given interval. This limit is $F(X_0, X)$ and is dependent upon X and X_0.

If we represent $F_n(X, X_0)$ as a summation we have

$$13) \qquad F_n(X, X_0) = \sum_{0}^{n-1} (X - X_{n-1}) f(X_{n-1}).$$

Then since $F(X, X_0)$ is equal to the limit of $F_n(X, X_0)$ as n becomes infinitely large we have,

$$14) \qquad \lim_{n=\infty} \sum_{0}^{n-1} (X - X_{n-1}) f(X_{n-1}) = \lim_{n=\infty} F_n(X, X_0)$$
$$= F(X, X_0).$$

Instead of the left hand member of the above equality, we may represent the limit of this sum by the symbol

$$\int_{X_0}^{X} f(X)\, dx$$

which indicates that the limit has been taken, of the sum indicated above, from X_0 to X. The values X and X_0 are called the upper and lower limits, respectively; and the value $\int_{X_0}^{X} f(X) dx$ is called a definite integral.

To complete the existence proof, it will be necessary to show that $F(X, X_0)$ is an integral of $f(X)$. To do this we must show that the derivative of $F(X, X_0)$ with respect to X is equal to $f(X)$.

We have always

$$\phi_n(X, X_0) \leq F(X, X_0) \leq \psi_n(X, X_0)$$

If g is the smallest and G the largest

values of $f(x)$ in the interval (x_0, x) we have

15) $\varphi_n(x, x_0) = g_0(x_1 - x_0) + g_1(x_2 - x_1) + \cdots + g_n(x - x_{n-1})$.

$\geq g(x_1 - x_0) + g(x_2 - x_1) + \cdots\cdots\cdots$

$\geq g(x - x_0)$,

and in the same way

16) $\psi_n(x, x_0) \leq G(x - x_0)$.

Then we will have

17) $g(x - x_0) \leq F(x, x_0) \leq G(x - x_0)$.

Let $x_0 = x$ and $x = x + h$. Then

18) $g \cdot h \leq F(x+h, x) \leq G \cdot h$,

where g is the smallest and G, the largest value of $f(x)$, in the interval $(x, x+h)$.

Then form the equation

19) $$\frac{F(x+h, x_0) - F(x, x_0)}{h} = \frac{F(x+h, x)}{h}$$

Now it follows from (18) and (19) that

20) $$g \leq \frac{F(x+h, x_0) - F(x, x_0)}{h} \leq G.$$

When $h = 0$, we have

21) $G = g = f(x)$.

Hence,

22) $$\lim_{h \doteq 0} \frac{F(x+h, x_0) - F(x, x_0)}{h} = f(x).$$

from (20) and (21) and from Theorem VI, page 20. Hence $f(x)$ has been shown to be the derivative of $F(x, x_0)$. We may state the

result in the following theorem.

Theorem II. Let the function $f(x)$ be positive and continuous in a definite interval (X_0, X), X_0 and x being its values within this interval and $x > X_0$. Then there is always one and only one continuous function $F(x, x_0)$ which disappears for $x = x_0$ and whose derivative with respect to x equals $f(x)$. This is given through the equation

$$F(x, x_0) = \lim_{n \doteq 0} \left\{ f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \quad + f(x_{n-1})(x - x_{n-1}) \right\}$$

in which $x_1 < x_2 < - - - - - < x_{n-1}$ every where represent values between x_0 and x.

As an example which illustrates this method of finding the definite integral, we have the following.

Ex. 1. $\int_a^b x^m \, dx$.

Since the value of the integral is independent of the manner of division of the interval from a to b, we let the points of division, x_1, x_2, x_3 - - - - - x_{n-1}, be such as to form with a and b, a geometric progression. Let $q = \sqrt[n]{\frac{b}{a}}$ so that $aq^n = b$. Then the values in question i.e. x_1, x_2, x_3, - - - - will be a, aq, aq^2, aq^3 - - - - aq^{n-1}; and the intervals are $a(q-1)$, $aq(q-1)$, $aq^2(q-1)$ - - - - $aq^{n-1}(q-1)$.

Then we have

1.) $S = a(q_r - 1)a^{m} + aq_r(q_r - 1)(aq_r)^{m} + aq_r^2(q_r - 1)(aq_r^2)^{m}$
$+ aq_r^3(q_r - 1)(aq_r^3)^{m} + \text{----} + aq_r^{n-1}(q_r - 1)(aq_r^{n-1})^{m}$.
$= a^{m+1}(q_r - 1)[1 + q_r^{m+1} + q_r^{2m+2} + q_r^{3m+3} + \text{---} + q_r^{m(n-1)+n-1}]$.

The expression

2.) $[1 + q_r^{m+1} + q_r^{2m+2} + \text{----} + q_r^{m(n-1)+n-1}]$,

is a geometric progression where

$a = 1, \quad r = q_r^{m+1}, \quad l = q_r^{m(n-1)+n-1}$.

Then the sum of (1) will be

3.) $S_1 = \dfrac{q_r^{m+1} \cdot q_r^{m(n-1)} - 1}{q_r^{m+1} - 1} = \dfrac{q_r^{mn+n} - 1}{q_r^{m+1} - 1}$.

But since $aq_r^n = b$, then $b^{m+1} = a^{m+1} q_r^{mn+n}$ and hence $q_r^{mn+n} = \dfrac{b^{m+1}}{a^{m+1}}$. Therefore we have from (3)

$$\frac{q_r^{mn+n} - 1}{q_r^{m+1} - 1} = \frac{\frac{b^{m+1}}{a^{m+1}} - 1}{q_r^{m+1} - 1} = \frac{b^{m+1} - a^{m+1}}{a^{m+1}(q_r^{m+1} - 1)}.$$

Hence from (1) we have

4.) $S = a^{m+1}(q_r - 1)\left[\dfrac{b^{m+1} - a^{m+1}}{a^{m+1}(q_r^{m+1} - 1)}\right] = \dfrac{q_r - 1}{q_r^{m+1} - 1} \cdot (b^{m+1} - a^{m+1})$.

It only remains to prove our division legitimate. We have to show that each of our intervals $a(q_r - 1)$, $aq_r(q_r - 1)$, ---- ---$aq_r^{n-1}(q_r - 1)$ approaches the limit zero as n increases indefinitely. Since

$$q_r^{\,n} = \frac{b}{a},$$

the limiting value of q_r, as n increases, must be 1, otherwise

$$\lim_{n=\infty} q_r^{\,n}$$

would not be finite. Therefore we have

$$\lim_{n=\infty} [a q_r^{\,k}(q_r - 1)] = \lim_{q_r = 1} [a q_r^{\,k}(q_r - 1)] = 0.$$

Now we have

5.) $$\int_a^b x^m dx = \lim_{n=\infty}(S) = \lim_{q_r = 1}\left[\frac{q_r - 1}{q_r^{m+1} - 1} \cdot (b^{m+1} - a^{m+1})\right]$$

$$= (b^{m+1} - a^{m+1}) \lim_{q_r = 1}\left(\frac{q_r - 1}{q_r^{m+1} - 1}\right).$$

But

$$\frac{q_r - 1}{q_r^{m+1} - 1} = \frac{1}{q_r^{m} + q_r^{m-1} + q_r^{m-2} \cdots + 1}$$

and

$$\lim_{q_r = 1}\left(\frac{q_r - 1}{q_r^{m+1} - 1}\right) = \lim_{q_r = 1} \frac{1}{q_r^{m} + q_r^{m-1} \cdots + 1} = \frac{1}{m+1}.$$

Therefore we have

$$\lim_{n=\infty} S = \frac{b^{m+1} - a^{m+1}}{m+1} = \int_a^b x^m dx.$$

In the preceding pages the definition of a definite integral was shown to hold for a continuous function. It now remains to show that this same definition will hold for more general functions. We shall use the following theorem to show this.

Theorem III. The necessary and sufficient condition that $f(x)$ has a definite integral in the interval (α, β), is that the lower limit of Δ_{n-1} is zero, where

$$\Delta = \sum_{0}^{n-1} (x_n - x_{n-1})\, \delta(x_n, x_{n-1}),$$

$\delta(x_n, x_{n-1})$ being the oscillation of the function in the interval (x_n, x_{n-1}).

To show the condition sufficient, i.e. when this condition exists the function may be integrated, we proceed as follows. Let the entire interval from x_0 to x_n be divided into sub-intervals in any way we please. Suppose the points of the division $x_1, x_2, x_3, \dots\dots x_{n-1}$ are such that the sub-intervals will be less than some δ_1. Now divide each of these sub-divisions into any number of intervals such that the points of the first division are all contained in the second and so on, and such that each of the sub-intervals will be divided into new sub-intervals each less than δ_2 etc. Continue this process an infinite number of times. We have restricted the values of the successive δ's in such a way that

1.) $$\lim_{n=\infty} \delta_n = 0.$$

Let G_n represent the maximum value of $f(x)$ in the interval (x_n, x_{n-1}), and let g_n represent the minimum value of $f(x)$ in the same interval. Now form the two sums,

2) $\sum_0^{n-1}(x_n - x_{n-1})G_n$, $\sum_0^{n-1}(x_n - x_{n-1})g_n$.

and denote their values by A_n and B_n. In the series of numbers $A_1, A_2, A_3 \cdots A_n$, each is less than, or at most equal to, the preceding, for, since G_n was taken as the largest value of $f(x)$ in the interval (x_n, x_{n-1}), in none of the sub-divisions of this interval can there be a greater value of $f(x)$. In the same way $B_1, B_2 B_3 \cdots B_n$ is an increasing series. Hence each of these two series,

$$A_1, A_2, A_3, A_4, - - - - - A_n$$
$$B_1, B_2, B_3, B_4, - - - - - B_n,$$

are monotone and consequently have limiting values. We must also notice that each A is always greater than any B. Because of these facts, the limiting values become identical when a place, n, can be found, for which, and for all values greater, the difference

$$A_n - B_n$$

becomes arbitrarily small and hence

$$\lim_{n \doteq \infty} (A_n - B_n) = 0 .$$

This is nothing else than our condition for the existence of the definite integral, for

$$\lim_{n \doteq \infty} (A_n - B_n) = \lim_{n \doteq \infty} \sum_{0}^{n-1} (x_n - x_{n-1})(G_n - g_n),$$

where

$$(G_n - g_n) = \delta(x_n, x_{n-1}).$$

The sum

$$S = (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \cdots + (x_n - x_{n-1}) f(x_{n-1}),$$

always lies between the limiting values of A_n and B_n and when these are equal the sum S has also the same finite determinate value. That is

$$A_n > S > B_n$$

$$\lim_{n \doteq \infty} A_n = \lim_{n \doteq \infty} B_n = A,$$

where A is some definite finite number. Then $\lim_{n \doteq \infty} S$ exists and equals A from Th. VI, page 20. We have shown that the function $f(x)$ has a definite derivative when the given condition exists because

$$\lim_{n \doteq \infty} S = \int_{x_0}^{x_n} f(x)\, dx$$

by definition.

It now remains to show that the condition is necessary; i.e. when the function may be integrated, the given condition

exists. If both series of numbers A_n and B_n had not the same limiting value, then the limiting value of S, where S is defined as above, would not be a determinate number. (Or, in other words, this single definite limit would not exist because S lies always between the two series and might have several values. We would have, if

$$\lim_{n \doteq \infty} (A_n - B_n) \neq 0$$

$$\lim_{n \doteq \infty} S = (?)$$

But this is contrary to our hypothesis and hence the condition must exist.

Remark. The definite integral may be described as the difference between the values of the indefinite integral formed for the upper and lower limits.

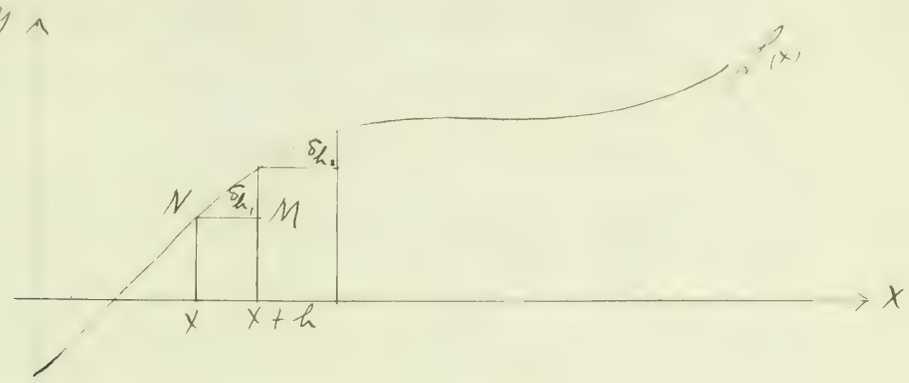
To show that the above remark is true we proceed as follows. From the law of the mean[1] we have

$$1) \qquad \frac{F(x+h) - F(x)}{h} = f(x + \theta h) \quad ; \; 0 < \theta < 1,$$

where $f(x + \theta h)$ is the mean value of $f(x)$ in the interval $(x, x+h)$. In this discussion

[1] See Dini § 72, 6.

$f(x)$ is considered to be finite and continuous throughout the interval (x_0, x_n). If we have



$f(x+\Theta h)$ as this mean value and if we write (1) as follows

2.) $$F(x+h) - F(x) = h \cdot f(x+\Theta h) \quad ; \; 0 < \Theta < 1,$$

the right hand side of (2) is nothing but the area between the x- axis, the curve and the ordinates $f(x)$ and $f(x+h)$. Now, if we form the rectangle x N M x+h, we may represent the remaining portion of the entire area by δ_{h_1}. Now we may write

3.) $$F(x+h) - F(x) = h\, f(x) \pm \delta_{h_1}.$$

If we place

$$x = x_0, \quad x+h = x_1, \quad h = (x_1 - x_0),$$

we may write instead of (3)

4.) $$F(x_1) = F(x_0) + (x_1 - x_0) f(x_0) \pm \delta_{h_1}$$

If we take a series of points chosen in any way we please, thus forming a series of intervals $h_1, h_2, h_3 \cdots h_n$, and a series of values $\delta_{h_1}, \delta_{h_2} \cdots \delta_{h_n}$, we may form the

following equations.

$$(5.)\quad F(x_1) = F(x_0) + (x_1 - x_0) f(x_0) \pm \delta_{h_1}$$
$$F(x_2) = F(x_1) + (x_2 - x_1) f(x_1) \pm \delta_{h_2}$$
$$F(x_3) = F(x_2) + (x_3 - x_2) f(x_2) \pm \delta_{h_3}$$
$$\text{- - - - - - - - - - - - - - - - - -}$$
$$F(x_n) = F(x_{n-1}) + (x_n - x_{n-1}) f(x_{n-1}) \pm \delta_{h_n}.$$

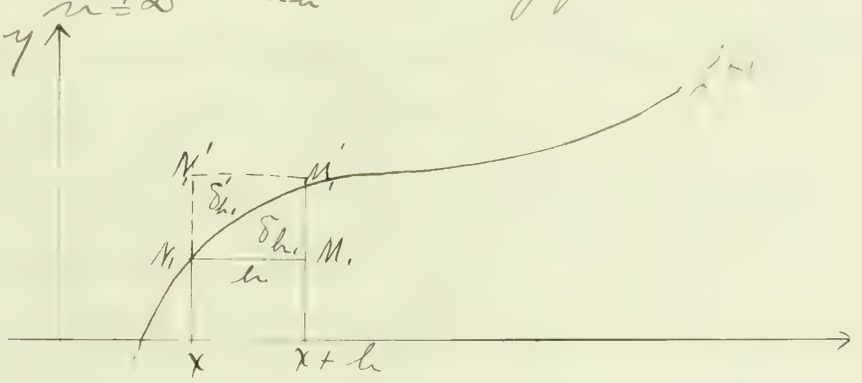
Now adding the above equations we have

$$(6.)\quad F(x_n) - F(x_0) = \Sigma (x_n - x_{n-1}) f(x_{n-1}) \pm \Sigma \delta_{h_n}.$$

Let n approach infinity and we have

$$(7.)\quad \lim_{n \doteq \infty} [F(x_n) - F(x_0)] = \lim_{n \doteq \infty} \Sigma (x_n - x_{n-1}) f(x_{n-1}) + \lim_{n \doteq \infty} \Sigma \delta_{h_n}.$$

We must now investigate the value of the expression $\lim_{n \doteq \infty} \Sigma \delta_{h_n}$. The difference between



the areas of the circumscribed rectangle, $x N' M' x+h$, and that of the inscribed rectangle, $x N M x+h$, is equal to $\delta'_h + \delta_h$ or to the small rectangle $N N' M' M$. Now, if we let G be the upper limit of all the values of $f(x)$ in the interval $(x, x+h)$ and g the lower limit in the same interval, the value of our small rectangle $N N' M' M$, is equal to $(G - g)h$. As the number

of divisions of our interval from x_0 to x_n becomes infinitely large the value of h becomes infinitely small. If we let $n = \infty$, then we have

$$\lim_{h=0} (G - g)\, h = 0$$

But since

$$(G - g)\, h = \delta'_{h_1} + \delta_{h_1},$$

then

$$\delta_{h_1} < (G - g)\, h$$

and

$$\lim_{h=0} \delta_{h_1} = 0.$$

Therefore since each of these small areas becomes infinitely small as h approaches zero, we would have

$$\lim_{h=0} \Sigma \delta_{hn} = 0 \qquad \lim_{n=\infty} \Sigma \delta_{hn} = 0.$$

Then from (7) we have

$$\lim_{n=\infty} [F(x_n) - F(x_0)] = \lim_{n=\infty} \Sigma (x_n - x_{n-1}) f(x_{n-1}) + \lim_{n=\infty} \Sigma \delta_{hn},$$
$$= \lim_{n=\infty} \Sigma (x_n - x_{n-1}) f(x_{n-1}) + 0$$

and

8) $$F(x_n) - F(x_0) = \lim_{n=\infty} \Sigma (x_n - x_{n-1}) f(x_{n-1}).$$

But the right hand side of this equation is nothing but the definite integral. Hence we have

$$F(x_n) - F(x_0) = \int_{x_0}^{x_n} f(x)\, dx,$$

and the demonstration is complete.

Theorem IV. The value of the definite integral

$$\int_a^b f(x)\,dx.$$

is equal to the product of the difference between the limits, by the value of the function $f(x)$, corresponding to a certain value $x=(\xi_1)$ between the limits of integration, i. e.

$$\int_a^b f(x)\,dx = (b-a)\,f(\xi_1)$$

Since the value of the definite integral does not depend upon the manner of division of the given interval, we may divide this into n equal parts. Then we have

$$1.)\quad \int_a^b f(x)\,dx = \lim_{\Delta x \doteq 0} \left[\Delta x \left\{ f(a) + f(a+\Delta x) + f(a+2\Delta x) + \cdots + f(b-\Delta x)\right\}\right],$$

where $\Delta x = \frac{b-a}{n}$. We may write

$$2.)\quad \int_a^b f(x)\,dx = \lim_{n \doteq \infty} \left[(b-a)\left\{\frac{f(a) + f(a+\Delta x) + f(a+2\Delta x) + \cdots + f(b-\Delta x)}{n}\right\}\right]$$

Now there must be some point ξ_1 in the interval, for which $f(\xi_1)$ is equal in value to the mean of the values of the different functions of x. The function must take this value $f(\xi_1)$ at least once throughout the interval because $f(x)$ is a continuous function. This value $f(\xi_1)$ will necessarily lie between the maximum and minimum values of $f(x)$, i. e.

$$m < f(\xi_1) < M,$$

where m is the minimum value of the function in the given interval and where M is the maximum value. Since there are to be n values of the function taken into consideration and since n increases indefinitely, the mean value will be reached through the limit. Hence we may define $f(\S_1)$ as follows, viz.

$$f(\S_1) = \lim_{n \doteq \infty} \left\{ \frac{f(a) + f(a+\Delta x) + \cdots + f(b-\Delta x)}{n} \right\}.$$

Now from (2) we have

$$\begin{aligned} \int_a^b f(x)\,dx &= \lim_{n \doteq \infty} \left[(b-a) \left\{ \frac{f(a) + f(a+\Delta x) + \cdots + f(b-\Delta x)}{n} \right\} \right] \\ &= (b-a) \lim_{n \doteq \infty} \left\{ \frac{f(a) + f(a+\Delta x) + \cdots + f(b-\Delta x)}{n} \right\} \\ &= (b-a)\, f(\S_1). \end{aligned}$$

and our theorem is proved.

Theorem V. <u>The definite integral is a continuous function of its upper limit.</u>

If we take the definite integral,

$$\int_\alpha^x f(x)\,dx,$$

for all values of x between α and β, and if this integral equals zero for $x = \alpha$, then the integral may be considered as a function of its upper limit, since it has, for all values of x between α and β, a definite finite value.

It now remains to show that this is a continuous function. Let us select $x \pm h$ as a point between α and β, right or left of x. Then we have

1.) $$\int_{\alpha}^{x \pm h} f(x)\,dx = \lim_{n \doteq \infty} [(x_1 - \alpha) f(\alpha) + (x_2 - x_1) f(x_1) + \cdots + (x + x_{n-1}) f(x_{n-1}) + (\xi_1 - x) f(x) + (\xi_2 - \xi_1) f(\xi_1) + \cdots + (x \pm h - \xi_n) f(\xi_n)]$$

where $(\xi_1 + \xi_2 + \cdots + \xi_n) = h$.

Also

2.) $$\int_{\alpha}^{x} f(x)\,dx = \lim_{n \doteq \infty} [(x_1 - \alpha) f(\alpha) + (x_2 - x_1) f(x_1) + \cdots + (x - x_{n-1}) f(x_{n-1})].$$

Subtract (2) from (1) and we have

3.) $$\int_{\alpha}^{x \pm h} f(x)\,dx - \int_{\alpha}^{x} f(x)\,dx = \lim_{n \doteq \infty} [(\xi_1 - x) f(x) + (\xi_2 - \xi_1) f(\xi_1) + \cdots + (x \pm h - \xi_n) f(\xi_n)].$$

This last is nothing but

$$\int_{x}^{x \pm h} f(x)\,dx.$$

By the last theorem we have

4.) $$\int_{x}^{x \pm h} f(x)\,dx = (x \pm h - x) f(\bar{\xi}_1) = h \cdot f(\bar{\xi}_1),$$

where $\bar{\xi}_1$ is some particular value between x and $x \pm h$, such that it is a mean value of all the other values of the function in this interval. Since the point $x \pm h$ is chosen arbitrarily, we can make h as small as we please. Consequently we may write

5.) $$\lim_{h \doteq 0} \left[\int_{\alpha}^{x \pm h} f(x)\,dx - \int_{\alpha}^{x} f(x)\,dx\right] = \lim_{h \doteq 0} h \cdot f(\bar{\xi}_1)$$

or since $f(\bar{\xi}_1)$ is a constant we may write

$$\lim_{h \doteq 0} \left[\int_a^{x+h} f(x)\,dx - \int_a^x f(x)\,dx \right] = f(\bar{\xi}_1) \lim_{h \doteq 0} h$$
$$= 0$$

Hence

6) $$\lim_{h \doteq 0} \int_a^{x+h} f(x)\,dx = \int_a^x f(x)\,dx.$$

But this is nothing but the condition for the continuity of a function. Hence the definite integral has been shown to be a continuous function of its upper limit. It is perfectly legitimate to talk about the limit in this case, because, as h is given different values which approach zero, the points $x \pm h$ become dense at the point x.

We have seen that the definite integral is equal to the difference between the values of the indefinite integrals formed for the upper and lower limits. From Theorem V we know that a definite integral is a continuous function of its upper limit. In the same way it could be shown that it is also a continuous function of its lower limit. Since these statements are true, we may write the definite integral as follows.

$$\int_a^b f(x)\,dx = \lim_{x \doteq b} F(x) - \lim_{x \doteq a} F(x),$$

where $F(x)$ is the indefinite integral of $f(x)$.

We should also consider the question of the existence of a definite integral whose upper limit becomes infinitely large. We may define a definite integral as follows.

$$\lim_{\epsilon \doteq 0} \int_{\alpha}^{\beta \pm \epsilon} f(x)\,dx = \int_{\alpha}^{\beta} f(x)\,dx.$$

Now, if we have an upper limit ω which becomes infinitely large, we may write

$$\lim_{\omega \doteq \infty} \int_{\alpha}^{\beta \pm \omega} f(x)\,dx = F(\alpha) - F(\alpha) = \int_{\alpha}^{\omega} f(x)\,dx.$$

In this sense, then, a definite integral whose upper limit becomes infinitely large exists and has a definite value.

Theorem VI. Interchanging the order of the limits of a definite integral changes the sign of the integral.

Since we have from above

$$\int_{a}^{b} f(x)\,dx = \lim_{x \doteq b} F(x) - \lim_{x \doteq a} F(x)$$

and

$$\int_{b}^{a} f(x)\,dx = \lim_{x \doteq a} F(x) - \lim_{x \doteq b} F(x),$$

then

$$\int_{b}^{a} f(x)\,dx = -\left[\lim_{x \doteq b} F(x) - \lim_{x \doteq a} F(x)\right]$$
$$= -\int_{a}^{b} f(x)\,dx.$$

and the theorem is proved.

Theorem VII. If c be a number between a and b in value, i.e. $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If

1) $$\int_a^b f(x)\,dx = \underset{x=b}{L} F(x) - \underset{x=a}{L} F(x)$$

and

2) $$\int_a^c f(x)\,dx = \underset{x=c}{L} F(x) - \underset{x=a}{L} F(x)$$

3.) $$\int_c^b f(x)\,dx = \underset{x=b}{L} F(x) - \underset{x=c}{L} F(x),$$

then by adding (2) and (3) we have (1).

Therefore

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Theorem VIII. If the functions $f_1(x), f_2(x), f_3(x) \cdots\cdots f_m(x)$ are finite and have definite integrals between α and β, and if there is a finite number of these functions then

$$f_1(x) \pm f_2(x) \pm f_3(x) + \cdots\cdots \pm f_m(x)$$

also finite and integrable between α and β.

We have from page 97 the following

$$\int_\alpha^\beta f_1(x)\,dx = \underset{x=\beta}{L} F_1(x) - \underset{x=\alpha}{L} F_1(x)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\int_\alpha^\beta f_m(x)\,dx = \underset{x=\beta}{L} F_m(x) - \underset{x=\alpha}{L} F_m(x).$$

Adding these equalities we have

1) $\int_\alpha^\beta f_1(x)\,dx + \int_\alpha^\beta f_2(x)\,dx + \int_\alpha^\beta f_3(x)\,dx + \cdots + \int_\alpha^\beta f_m(x)\,dx =$

$$\sum_{x=\beta} F_1(x) + \sum_{x=\beta} F_2(x) + \sum_{x=\beta} F_3(x) + \cdots + \sum_{x=\beta} F_m(x) - \sum_{x=\alpha} F_1(x) -$$

$$\sum_{x=\alpha} F_2(x) - \sum_{x=\alpha} F_3(x) - \cdots - \sum_{x=\alpha} F_m(x).$$

Now we may write the right hand member of (1) as follows, viz.

2) $\left[\sum_{x=\beta} F_1(x) + \sum_{x=\beta} F_2(x) + \cdots + \sum_{x=\beta} F_m(x) - \sum_{x=\alpha} F_1(x) - \sum_{x=\alpha} F_2(x) - \cdots - \sum_{x=\alpha} F_m(x)\right]$

$$= \sum_{x=\beta}\left[F_1(x) + F_2(x) + \cdots + F_m(x)\right] - \sum_{x=\alpha}\left[F_1(x) + F_2(x) + \cdots + F_m(x)\right].$$

We may do this since there are a finite number of terms. But the right hand side of (2) is nothing but

$$\int_\alpha^\beta [f(x_1) + f(x_2) + f(x_3) + \cdots + f_m(x)]$$

and we have proved the theorem.

Theorem IX. *If a finite number of functions*

$f_1(x),\ f_2(x),\ f_3(x) \cdots f_m(x)$

are integrable between α and β then their product

$f_1(x) \cdot f_2(x) \cdot f_3(x) \cdots f_m(x)$

is also finite and integrable.

If we place D = the oscillation of the

product and D_1, D_2, the oscillation of $f_1(x)$ and $f_2(x)$. and let L_1 and L_2 be the upper limits and l_1 and l_2, the lower limits of $f_1(x)$ and $f_2(x)$, in the same interval, then

$$D \leqq L_1 L_2 - l_1 l_2$$

and

$$D \leqq L_1 (L_2 - l_2) + l_2 (L_1 - l_1)$$

or

$$D \leqq L_1 D_2 + l_2 D_1 .$$

Now if $f_1(x)$ and $f_2(x)$ are always positive and the condition for integration is satisfied, then is their product integrable for

$$\lim_{n \doteq \infty} D \leqq \lim_{n \doteq \infty} \{ L_1 D_2 + l_2 D_1 \}.$$

$$\leqq \lim_{n \doteq \infty} L_1 D_2 + \lim_{n \doteq \infty} l_2 D_1$$

$$= 0,$$

from the condition for the existence of a definite integral. where n is the number of sub-intervals into which (α, β) is divided. In the same way we could show that the product of any finite number of such terms will be integrable.

Theorem X. The quotient $\frac{f(x)}{\phi(x)}$ of two functions $f(x)$ and $\phi(x)$, which are, in a

finite interval (α, β) always finite and integrable, is in the same interval integrable, if the number $\phi(x)$ is constantly finite between α and β and differs from zero.

If D, D_1, D_2 are the respective oscillations of the functions $\frac{f(x)}{\phi(x)}$, $f(x)$ and $\phi(x)$ in a given sub-interval, and if l_1, l_2 are the lower limits of $f(x)$ and $\phi(x)$, while L_1 and L_2 are the upper limits of $f(x)$ and $\phi(x)$, then, for a positive l_2 and L_2, we have

$$D \leq \frac{L_1}{l_2} - \frac{l_1}{L_2} \quad \text{or} \quad D = \frac{L_1 D_2 + l_2 D_1}{l_2 L_2}.$$

Now from the condition given for the existence of a definite integral and since we have assumed that $f(x)$ and $\phi(x)$ may be integrated, we may divide up the given interval into a finite number of sub-intervals such that the oscillation in each sub-interval is less than an arbitrarily small number σ. Hence we may write

$$D \leq \frac{L_1 + l_2}{l_2 L_2} \sigma.$$

Since, as was stated above, σ may be made

as small as we please, we may write

$$\lim_{\sigma \doteq 0} D \leq \lim_{\sigma \doteq 0} \frac{L_1 + L_2}{l_2 L_2} \cdot \sigma.$$
$$= 0$$

Then since the lower limit of the oscillation of the function $\frac{f(x)}{\phi(x)}$ is equal to zero, the condition for the existence of the definite integral is fulfilled. Hence the theorem is proved.

Theorem XI If the function $f(x)$ has a definite integral within a given interval (α, β) and if its maximum value M and its minimum value m are of the same signs, then $\frac{1}{f(x)}$ will also be integrable in the given interval.

Let D_k be the oscillation of $f(x)$ in the interval $(x_k - x_{k-1})$, and D_k', the oscillation of $\frac{1}{f(x)}$ in the same interval. Then

$$D_k' = \left| \frac{1}{m_k} - \frac{1}{M_k} \right| = \left| \frac{M_k - m_k}{M_k m_k} \right|.$$

Since m_k is less than M_k, we have

$$\left| \frac{M_k - m_k}{M_k m_k} \right| \leq \left| \frac{M_k - m_k}{m_k^2} \right| = \frac{D_k}{m_k^2}$$

Then

$$\lim_{n \doteq \infty} D'_K \leq \lim_{n \doteq \infty} \frac{D_K}{m_K} = 0.$$

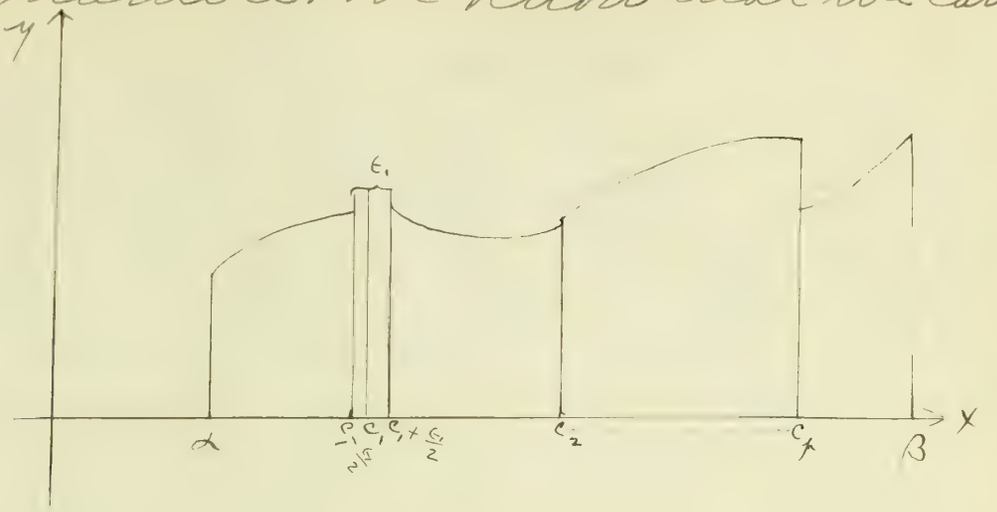
Since the lower limit of the oscillation of the function $\frac{1}{f(x)}$ is zero, then the condition for the existence of a definite integral is satisfied and our theorem is demonstrated.

The special cases of functions which may have definite integrals are considered in the following theorems.

Theorem XII. The finite functions which are continuous in general between α and β, i.e. which have discontinuities at a finite number of points, and those point wise discontinuous functions whose discontinuities occur at points which form an infinitely great assemblage of the first order, may have definite integrals.

Let us first consider those functions which have only a finite number of discontinuities. Suppose the function $f(x)$ is finite and continuous throughout the interval (α, β) but has discontinuities at the points $c_1, c_2 - - - - - c_p$, which

are finite in number. We know that we can



inclose each of the c points in an interval which may be made arbitrarily small. Let

$$\epsilon_1, \epsilon_2, \epsilon_3 \quad \cdots\cdots \epsilon_p$$

be the extent of each of these intervals. Since we have defined a definite integral as

$$\int_\alpha^\beta f(x)\,dx = \lim_{n=\infty} \sum_0^{n-1} (x_n - x_{n-1})\, f(x_{n-1}),$$

and since we represented by this expression the area between the x-axis, the curve and the ordinates corresponding to the limits in question, we may represent the given area in this case in a similar manner. It will be seen that the area in question will differ from that in which $f(x)$ is considered as a continuous function by an amount which is equal to the sum of the small rectangles

at each of the c points. These values will in some cases be added and in others subtracted from the original area. If we consider these areas equal to the value of $f(x)$ at the c points, times the extent of the interval, as they will be if the intervals are chosen small enough, we may write

1.) $$\int_{\alpha}^{\beta} f(x)\,dx = \lim_{n=\infty} S = \lim_{n=\infty}\left\{(x_1-\alpha)f(\alpha)+(x_2-x_1)f(x_1)+\cdots\right.$$
$$\left.\cdots\cdots+(\beta-x_{n-1})f(x_{n-1})\right\} \pm \lim_{\epsilon=0}\left\{\epsilon_1 f(c_1)+\epsilon_2 f(c_2)+\cdots\right.$$
$$\left.+\cdots\cdots+\epsilon_p f(c_p)\right\}.$$

It should be noted that as n becomes infinitely large each of the ϵ's become infinitely small. The value ϵ taken above for the limit is the upper limit of all the values of ϵ. Since the ϵ intervals were chosen arbitrarily small we have

2.) $$\lim_{\epsilon=0}\left\{\epsilon_1+\epsilon_2+\epsilon_3+\cdots\cdots+\epsilon_p\right\}=0.$$

where ϵ has the same meaning as above. If we take $f(c)=$ the upper limit of the values of the function in the ϵ intervals we may write

3.) $$\lim_{n=\infty} S = \lim_{n=\infty}\left\{(x_1-\alpha)f(\alpha)+(x_2-x_1)f(x_1)+\cdots+(\beta-x_{n-1})f(x_{n-1})\right\}$$

$$\pm \lim_{\epsilon=0}\left\{(\epsilon_1+\epsilon_2+\epsilon_3+\cdots+\epsilon_p)f(c)\right\}.$$
$$= \lim_{n=\infty}\left\{(x_1-\alpha)f(\alpha)+(x_2-x_1)f(x_1)+\cdots+(\beta-x_{n-1})f(x_{n-1})\right\}.$$

from (2). Therefore we have

$$\lim_{n=\infty} S = \int_{\alpha}^{\beta} f(x)\, dx$$

and this part of the theorem is demonstrated.

If the function has a discontinuity at each of the points of an assemblage of the first order we may show in the same way that it may have a definite integral. Because we may always inclose the points of such an assemblage in intervals whose sum can be made to differ as little from zero as we please (1) We have as above

$$\lim_{\epsilon=0} (\epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots + \epsilon_m) = 0.$$

where ϵ is equal to the upper limit of the series $\epsilon_1, \epsilon_2, \epsilon_3, \cdots \epsilon_n$. Forming the limit of the sum which represents the area in question, we have the same result as expressed in (3). Hence the theorem is fully demonstrated. As a corollary to the above theorem we may state the following.

Cor. If the function $f(x)$ is finite and has a definite integral in the interval (α, β) and if we change the value of the function at a finite number of points, or if the value is changed at an infinitely large assemblage of points of the first

(1) See Dini §14, Page 26.

order, yet so that the function remains finite, then is the new function integrable between α and β and the value of the integral

$$\int_{\alpha}^{\beta} f(x)\,dx$$

does not change.

Let $\phi(x)$ be the new function. We can inclose the points at which the value of $f(x)$ has been changed in a finite number of intervals, $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_m$, whose sum is smaller than an arbitrary number ϵ_1. The remaining intervals in which the function has had no change may be divided into sub-intervals, $\delta_1, \delta_2 \cdots$, so that the sum of those intervals, in which the oscillations occur, which are larger than a given arbitrarily small and positive number τ, is smaller than ϵ_2. By this division of the entire interval, (α, β) into the sub-intervals $(\delta_1, \delta_2 \cdots, \lambda_1, \lambda_2 \cdots)$ the changed function $\phi(x)$ can only have oscillations greater than an arbitrary number τ in the intervals whose sum is less than $\epsilon_1 + \epsilon_2$. But since ϵ_1 and ϵ_2 may be chosen as small as we please we have

$$\lim_{n \doteq \infty} (\epsilon_1 + \epsilon_2) = 0 .$$

Then by the previous theorem the new function $\phi(x)$ may have a definite integral.

The new XIII. The functions which in the finite interval, are always finite and possess the property that the number of points at which discontinuities occur which are greater than an arbitrary positive number τ_1 is constantly finite and only by the unlimited decrease of τ can increase without measure are capable of having a definite integral.

Let the points $x_1, x_2, x_3 \ldots\ldots x_m$ be the points at which the amount of discontinuity is equal to or greater than τ_1 i.e. when $D \geq \tau_1$. We may inclose these points in the intervals $\delta_1, \delta_2, \delta_3 \ldots\ldots \delta_m$ which may be made arbitrarily small. Outside these there are a number of intervals $u_1, u_2, u_3 \ldots\ldots$ at which

$$D < \tau_1$$

Hence we have because τ may be chosen as small as we please

$$\lim_{\tau = 0} \text{Oscillations} = 0$$

for these last mentioned intervals.

Since we have m δ-intervals, each of which is arbitrarily small, and if D represents the largest amount of discontinuity among them we would have added to S a number not greater than $m\delta D$ where

$$S = (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) + \cdots + (x_n - x_{n-1}) f(x_{n-1}).$$

Since m and D are finite numbers we can so choose δ that

$$\lim_{\delta = 0} m \delta D = 0.$$

Then, as in Theorem XII, the function has a definite integral.

Theorem XIV. *Functions which are always finite in the finite interval (α, β) and have only points of discontinuity of the ordinary kind or such discontinuities of the second kind which occur always upon one and the same side, right or left of the points in question, are integrable between α and β.*

Let $f(x)$ be such a function. We will assume that, if it has a discontinuity of the second kind in one or more points of the interval, this discontinuity occurs only at the right of these points. Suppose

$\alpha L \beta$. Because the function is continuous or at most has a discontinuity of the first kind in the interval $(x, x+\epsilon)$, then there are an infinite number of points of continuity in this interval. If this were not true, the function would be totally discontinuous and the value of the function could have no limit existing within this interval which is contrary to hypothesis. At these points the function is continuous on both sides. If x_1 is one of these points, then we could so select an interval $(x_1, x_1+\epsilon_1)$ end points at most excluded, within which the values of x fulfill the following condition,

1.) $$\lim_{x \doteq x_1} f(x) = f(x_1).$$

There would be an infinite number of these ϵ_1's because if (1) is true for ϵ_1 it will be true for any number smaller than ϵ_1. We can select one which will be the upper limit of all the others.

Now place $y_1 = x_1 + \epsilon_1$ and form the interval

$$(y_1, \; y_1 + \tfrac{1}{2}\epsilon).$$

Select in this interval a point x_2 at the right of y_1 at which the function is con-

tinuous. Then we will form for the point x_2 an interval $(x_2,\ x_2 + \epsilon_2)$ in which as before in $(x_1,\ x_1 + \epsilon_1)$ for every point x, end points at most excluded, the following condition exists. i.e.

2) $$\underset{x \doteq x_2}{L} f(x) = f(x_2),$$

where as before ϵ_2 is the upper limit of the values which the ϵ's can take if for every x lying between x_2 and $x_2 + \epsilon_2$ ($x_2 + \epsilon_2$ excluded) the above condition holds. Now place $y_2 = x_2 + \epsilon_2$ and form the interval

$$(-y_2,\ y_2 + \tfrac{1}{2}\epsilon_2),$$

and take a new point x_3 at the right of y_2 in which the given function is continuous. Now form a new interval $(x_3, x_3 + \epsilon_3)$ similar to $(x_1,\ x_1 + \epsilon_1)$ etc. so that for every point of the same

$$\underset{x \doteq x_3}{L} f(x) = f(x_3).$$

Continue this[3] farther and we have a number system $x_1,\ x_2,\ x_3$ - - - - - that increases constantly. In this way one either arrives at

1.) The point β after a finite number of intervals; or

2.) The number of points $x_1,\ x_2,\ x_3$ - - - increases indefinitely and thus have a definite

limit which is equal to or less than B.

But case (2) can not be true. For if it were possible and A were the limit of the infinite series of values x_1, x_2, x_3 —— where $A \geq B$, then in every interval however small, at the left of A, an infinite by large number of these points occur. Now because $f(x)$ is continuous or at most has an ordinary discontinuity upon the left of the point A, then we can always determine an interval $(A-\omega, A)$ at the left of A, so that there must be an infinite number of points x_1, x_2, x_3 ——. Now if x_m were one of these points, then we has for every one of these points of the interval (at most A excluded) the condition

$$\underset{x \geq x_m}{L} \; f(x) = f(x_m).$$

Then since $\epsilon_1, \epsilon_2, \epsilon_3$ —— —— are definite finite numbers the end points of the interval $(x_m, x_m + \epsilon_m)$ can not fall within the interval $(A-\omega, A)$, but could at most coincide with A because of the manner in which the intervals are selected. The following point x_{m+1} would then fall beyond the interval $(A-\omega, A)$ which is a contradiction because A

was to be the limit of the points $x_1, x_2, x_3 \cdots$
We see then that there must be only a finite
number of intervals

$(x_1, x_1+\epsilon_1), (x_2, x_2+\epsilon_2), (x_3, x_3+\epsilon_3)$ - - - - -
by which the point β is reached. Now if
we take at the left of

$x_1+\epsilon_1, \quad x_2+\epsilon_2, \quad x_3+\epsilon_3, \; - - - - - - - -$

within the respective intervals

$(x_1, x_1+\epsilon_1), (x_2, x_2+\epsilon_2) - - - -$

the points $y_1', y_2', y_3' - - -$ which are
each less than

$x_1+\epsilon_1, \quad x_2+\epsilon_2 - - - - - - - - - -$

by $\frac{1}{2}\epsilon, \quad \frac{1}{2^2}\epsilon, \quad \frac{1}{2^3}\epsilon - - - - - - - - -$ where
ϵ has the same meaning as in the beginning.
We shall consider the divisions of the entire
interval (α, β) which correspond to the points

$\alpha, x_1, y_1', \quad x_2, \quad y_2', \quad x_3 - - - - - - -$. One sees
that this division gives rise to a finite
number of intervals

$(\alpha, x_1), \quad (x_1, y_1'), \quad (y_1', x_2), \quad (x_2, y_2') - - -$
and greater oscillations than σ can occur
only in the intervals

$(\alpha, x_1), \quad (y_1', x_2), \quad (y_2', x_3), \quad (y_3', x_4) - - -$
and these last are smaller respectively
than

$\epsilon, \quad \epsilon, \quad \frac{1}{2}\epsilon, \quad \frac{1}{2^2}\epsilon, \quad \frac{1}{2^3}\epsilon - - - -$

and their sum is less than 3ϵ. Since ϵ is an arbitrarily small number the sum of these intervals, 3ϵ, can be made to equal zero in the limit. i. e.

$$\lim_{\epsilon=0}(\epsilon+\epsilon+\tfrac{\epsilon}{2}+\tfrac{\epsilon}{2^n}+\epsilon+\text{----})=0.$$

Then by the method of Theorem XII page 104 we can show that the function has a definite integral.

Theorem XV. Functions which are totally discontinuous in the interval (α, β) cannot have a definite integral in this interval.

In this case, in each arbitrarily small interval either in the entire interval (α, β) or in some definite part of it, the discontinuity of the function is larger at every point than a definite number τ. Since the limit of the oscillation of the function in each of these sub-intervals is greater than zero, it is impossible for the condition given in Theorem II page 87, to exist. i. e. The lower limit of

$$\Delta \equiv \Sigma(x_n - x_{n-1})\, f(x_{n-1})\, \delta(x_n, x_{n-1}) \neq 0$$

for

$$\lim \delta(x_n, x_{n-1}) \neq 0.$$

Hence the theorem has been proved.

Theorem XVI. If $f(x)$ is a finite, integrable function between x_0 and x_n then the value of the integral

$$\int_{x_0}^{x_n} f(x)\, dx$$

is not changed if one changes the value of the function in an arbitrary assemblage of points of the first a second species provided that the new function $\varphi(x)$ is integrable between x_0 and x_n and in every sub-interval, however small, there are points in which the value of the function is unchanged.

The two functions differ only in a set of points which form an assemblage of the first a second species. The points of such an assemblage may be inclosed in a number of intervals whose sum may be used to differ from zero by as little as we please. Let these intervals be $u_1, u_2, u_3, \dots u_p$ such that

1.) $S_1 = \sum_{u=0}^{u=p} (u_1 + u_2 + u_3 + \dots + u_p) = 0$

where u is equal to a greater than the upper limit of all the u's. In the remaining interval $\delta_1, \delta_2 \dots \delta_m$ the values

of the two functions are equal. We would then have

$$2.)\ \Big[\sum_{1}^{n-1}(x_n - x_{n-1})\,f(x_{n-1}) - \sum_{1}^{n-1}(x_n - x_{n-1})\,\varphi(x_{n-1})\Big] =$$

$$\sum_{1}^{m}(x_m - x_{m-1})\Big[f(x'_{m-1}) - \varphi(x'_{m-1})\Big] - \sum_{1}^{p}\mu_p\Big[f(x)'_{p-1} - \varphi(x)'_{p-1}\Big]$$

Then

$$3.)\ \lim_{n\doteq\infty}\Big[\sum_{1}^{n-1}(x_n - x_{n-1})\,f(x'_{n-1}) - \sum_{1}^{n-1}(x_n - x_{n-1})\,\varphi(x'_{n-1})\Big] =$$

$$\lim_{n\doteq\infty}\sum_{1}^{m}(x_m - x_{m-1})\Big[f(x'_{m-1}) - \varphi(x'_{m-1})\Big] - \lim_{n\doteq\infty}\sum_{1}^{p}\mu_p\Big[f(x)'_{p-1} - \varphi(x)'_{p-1}\Big].$$

But the first part of the right hand member of (3) is zero because the value of the function has not changed in the δ intervals and hence $\Big[f(x'_{m-1}) - \varphi(x'_{m-1})\Big] = 0$. From (1) the second part of the right hand member is also zero. We then have

$$4.)\ \lim_{n\doteq\infty}\Big[\sum_{1}^{n-1}(x_n - x_{n-1})\,f(x'_{n-1})\Big] - \lim_{n\doteq\infty}\Big[\sum_{1}^{n-1}(x_n - x_{n-1})\,\varphi(x)'_{n-1}\Big] = 0$$

Since

$$\lim_{n\doteq\infty}\sum_{1}^{n-1}(x_n - x_{n-1})\,f(x_{n-1}) = \int_{x_0}^{x_m} f(x)\,dx$$

we have

$$\lim_{n\doteq\infty}\sum_{1}^{n-1}(x_n - x_{n-1})\,\varphi(x'_{n-1}) = \int_{x_0}^{x_m} f(x)\,dx$$

and the value of the integral is unchanged

Theorem XVII. If the known function $f(x)$ has the same sign at the points at which it is different from zero, then

is the integral

$$\int_{\alpha}^{\beta} f(x)\,dx$$

either zero or a number different from zero which has the same or opposite sign as $f(x)$ according as $\alpha < \beta$ or $\alpha > \beta$, and if moreover, there exists, between α and β an interval of finite extent in which $f(x)$ differs from zero by a definite number, then the integral

$$\int_{\alpha}^{\beta} f(x)\,dx$$

differs from zero.

Suppose $f(x)$ always positive and $\alpha < \beta$. Then since, by the discussion on page 97, we have

1.) $$\int_{\alpha}^{\beta} f(x)\,dx = \lim_{x \doteq \beta} f(x) - \lim_{x \doteq \alpha} f(x),$$

we would have the integral equal to either zero or a positive quantity. This is true because since $\beta > \alpha$ the limit

$$\lim_{x \doteq \beta} f(x)$$

must be larger than or at most equal to

$$\lim_{x \doteq \alpha} f(x).$$

Now by (2) the right hand side of (1) is either a positive number or zero. On the other hand, if $\alpha > \beta$, then we have

$$\lim_{x \doteq \beta} f(x) = \lim_{x \doteq \alpha} f(x)$$

and from (1) we have

$$\int_{\alpha}^{\beta} f(x)\,dx = 0$$ or a negative number.

If between α and β a finite interval exists in which $f(x) \neq 0$, we are to show that the integral is different from zero. Let (x_1, x_2) be the end points of such an interval. We have from Theorem VII page 99.

$$(1)\quad \int_{\alpha}^{\beta} f(x)\,dx = \int_{\alpha}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{\beta} f(x)\,dx.$$

$$= \left[\lim_{x \doteq x_1} f'(x) - \lim_{x \doteq \alpha} f'(x)\right] + \left[\lim_{x \doteq x_2} f'(x) - \lim_{x = x_1} f'(x)\right]$$

$$+ \left[\lim_{x \doteq \beta} f'(x) - \lim_{x \doteq x_2} f'(x)\right]$$

Now for every point in (x_1, x_2) the function is different from zero, but has the same sign. Then since $x_2 > x_1$, we have

$$\lim_{x \doteq x_2} f'(x) > \lim_{x \doteq x_1} f'(x)$$

from the general properties of limits. Then

$$(2)\quad \left[\lim_{x \doteq x_2} f'(x) - \lim_{x \doteq x_1} f'(x)\right] \neq 0.$$

From the first part of the theorem since $\alpha < \beta$, the integrals

$$\int_{\alpha}^{x_1} f(x)\,dx \quad \text{and} \quad \int_{x_2}^{\beta} f(x)\,dx$$

are either both zero or both positive

numbers. Then, no matter which of these cases are fulfilled, since (2) is true we have

$$\int_{\alpha}^{\beta} f(x)\,dx \neq 0$$

and the theorem is demonstrated.

Theorem XVIII. The function $\Psi(x)$, which is integrable between α and β, and whose integral throughout every arbitrary sub-interval of this interval (α, β) is constantly zero, must in infinitely many points of every arbitrary interval, however small, be always zero.

If this were not true, there would be in a sub-interval, always infinitely many points existing at which $\Psi(x)$ is continuous and for which

$$\Psi(x') = A \neq 0$$

where x' is any one of these points. There would be about x' a region $(x'-\epsilon, x'+\epsilon)$ for which

$$\Psi'(x') \neq 0$$

since the function is not constant. Then

$$\left[\underset{x'=x'+\epsilon}{L}\Psi'(x') - \underset{x'=x'-\epsilon}{L}\Psi'(x')\right] \neq 0.$$

Then the integral

$$\int_{x'-\epsilon}^{x'+\epsilon} \psi(x')\,dx \neq 0.$$

But since this is contrary to our hypothesis, $\psi(x')$ cannot differ from zero and the theorem is proved.

Theorem XIX. *If $f(x)$ is a finite, integrable function in the interval (x_0, x_n) then the function $f_1(x)$ which is formed by the absolute value of $f(x)$ is also integrable between the same limits.*

If $f_1(x) = |f(x)|$ then the oscillations of $f_1(x)$ can never be greater than those of $f(x)$ in the interval. Since the absolute value of a sum is equal to or less than the sum of the absolute values, we have

$$\left|\sum_1^{n-1}(x_n - x_{n-1})\,f(x_n),\right| \leq \sum_1^{n-1}|(x_n - x_{n-1}),\, f(x_{n-1})|$$

But the limit of the absolute values of a number is never less than the absolute value of the limit. Hence

$$\left|\lim_{n=\infty}\sum_1^{n-1}(x_n - x_{n-1})\,f(x_{n-1})\right| \leq \lim_{n=\infty}\sum|(x_n - x_{n-1})\,f(x_{n-1})|.$$

Since $f_1(x)$ was defined as $|f(x)|$, we have

$$\left|\lim_{n=\infty}\sum_1^{n-1}(x_n - x_{n-1})\,f(x_{n-1})\right| \leq \lim_{n=\infty}\sum_1^{n-1}(x_n - x_{n-1})\,f_1(x_{n-1})$$

or

$$\int_{x_0}^{x_n} f(x)\,dx \leqq \int_{x_0}^{x_n} f_1(x)\,dx.$$

and the theorem is proved.

Theorem XX. If the two functions $f(x)$ and $\varphi(x)$ are finite and continuous in the interval (α, β) and if $f(x) \geqq \varphi(x)$, then we have the condition

$$\int_{\alpha}^{\beta} f(x)\,dx \geqq \int_{\alpha}^{\beta} \varphi(x)\,dx$$

or

$$\int_{\alpha}^{\beta} f(x)\,dx \leqq \int_{\alpha}^{\beta} \varphi(x)\,dx$$

according as $\alpha < \beta$ or $\alpha > \beta$.

Since the integral

$$\int_{\alpha}^{\beta} f(x)\,dx = \lim_{n=\infty} \sum_{1}^{n-1} (\beta - x_{n-1})\, f(x_{n-1})$$

then we have, since $f(x)$ is always $\geqq \varphi(x)$

$$\lim_{n=\infty} \sum_{1}^{n-1} (\beta - x_{n-1})\, f(x_{n-1}) \geqq \lim_{n=\infty} \sum_{1}^{n} (\beta - x_{n-1})\, \varphi(x_{n-1}),$$

providing $\alpha < \beta$. But if $\alpha > \beta$, then we have by Theorem XVII, page 117, that $\int_{\alpha}^{\beta} f(x)\,dx$ is of opposite sign from $f(x)$ and $\int_{\alpha}^{\beta} \varphi(x)\,dx$ is of opposite sign from $\varphi(x)$. Then the value of

$$\lim_{n=\infty} \sum_{1}^{n-1} (\beta - x_{n-1})\, f(x_{n-1})$$

is still larger, numerically than

$$\lim_{n=\infty} \sum_{1}^{n-1} (\beta - x_{n-1})\, \varphi(x_{n-1}),$$

but if the sign before each becomes negative then we have the condition that

$$\lim_{n=\infty} \sum_{1}^{n-1} (\beta - x_{n-1})\, f(x_{n-1}) \leqq \lim_{n=\infty} \sum_{1}^{n-1} (\beta - x_{n-1})\, \varphi(x_{n-1}).$$

This completes the proof of the theorem.

Theorem XXI. Let the function $\phi(x)$ be finite and integrable between α and β $(\alpha < \beta)$, and let the same be true of $f(x)$, or at least, let the same be true of the product $\phi(x) \cdot f(x)$. Also let $f(x)$ always be numerically less than a finite number. If we build up a function $f_1(x)$ out of the absolute value of $f(x)$ and if L is the upper limit of the absolute values of $f(x)$ between α and β, or a larger number, then we have

$$\left| \int_\alpha^\beta f(x) \cdot \phi(x) \, dx \right| \leq L \int_\alpha^\beta \phi_1(x) \, dx.$$

If $f_1(x) = |f(x)|$ and $\phi_1(x) = |\phi(x)|$, then from Theorem XX, page 122, we have

$$\left| \lim_{n=\infty} \sum_{n-1}^{1} (\beta - x_{n-1}) f(x_{n-1}) \right| \leq \lim_{n=\infty} \sum_{n-1}^{1} (\beta - x_{n-1}) f_1(x).$$

Now if $f_1(x)$ is equal to $|f(x)|$, then surely the upper limit L_1 of $|f(x)|$ would not be smaller than $f_1(x)$, and we would have

$$\left| \int_\alpha^\beta f(x) \cdot \phi(x) \, dx \right| \leq \int_\alpha^\beta f_1(x) \cdot \phi_1(x) \, dx$$

$$\leq L \int_\alpha^\beta \phi_1(x) \, dx$$

and the theorem is established.

Theorem XXII. Let the function $\varphi(x)$ be finite and integrable between α and β and have nowhere it is different from zero, always the same sign. At the same time let the function $f(x)$ be always finite between α and β and, moreover, let it be integrable, or at least let the product $f(x)\,\varphi(x)$ be in this interval integrable. Indicate the upper and lower limit of the values of $f(x)$ between α and β by L and l respectively. Then the following condition holds, i.e.:

$$L\int_\alpha^\beta \varphi(x)\,dx \geqq \int_\alpha^\beta f(x).\,\varphi(x)\,dx \geqq l\int_\alpha^\beta \varphi(x)\,dx.$$

where $\varphi(x)$ is different from zero and is positive. (On the other hand, if $\varphi(x)$ is negative, where it is not zero, we have

$$L\int_\alpha^\beta \varphi(x)\,dx \leqq \int_\alpha^\beta f(x).\,\varphi(x)\,dx \leqq l\int_\alpha^\beta \varphi(x)\,dx.$$

Since L is the upper limit of $f(x)$ and l the lower limit, then

1.) $$\int_\alpha^\beta L.\,\varphi(x)\,dx \geqq \int_\alpha^\beta f(x).\,\varphi(x)\,dx$$

because $f(x)$ cannot have a value greater than L. Likewise $f(x)$ cannot have in the

interval (α, β) any value less than l.
and we have,

2.) $$\int_\alpha^\beta l \cdot \varphi(x)\,dx \leq \int_\alpha^\beta f(x) \cdot \varphi(x)\,dx.$$

Then since L and l are constants, we have

$$\int_\alpha^\beta L \cdot \varphi(x)\,dx = L\int_\alpha^\beta \varphi(x)\,dx \quad \text{and} \quad \int_\alpha^\beta l \cdot \varphi(x)\,dx = l\int_\alpha^\beta \varphi(x)\,dx.$$

Then

$$L\int_\alpha^\beta \varphi(x)\,dx \geq \int_\alpha^\beta f(x) \cdot \varphi(x)\,dx \geq l\int_\alpha^\beta \varphi(x)\,dx.$$

The above is true if $\varphi(x)$ is a positive number.

If $\varphi(x)$ is negative we have

$$\int_\alpha^\beta \varphi(x)\,dx = L \lim_{n=\infty} \sum^{n+1}{}' (\beta - x_{n-1})\,\varphi(x_{n-1}) = \text{a negative quantity. Then}$$

$$L\int_\alpha^\beta \varphi(x)\,dx \leq \int_\alpha^\beta f(x) \cdot \varphi(x)\,dx \leq l\int_\alpha^\beta \varphi(x)\,dx.$$

This completes the proof of the theorem.

§ 4. Applications of Definite Integrals

We shall discuss here, some of the particular applications of the definite integral. The ordinary text book in Integral Calculus treats these subjects from the stand point of infinitesimal quantities of the first, second, third and higher orders. The purpose of this discussion is to show that these same simple applications of the integral calculus may be based directly upon the properties of limits. In particular, the area of curves, volume of solids of revolution, lengths of curves, and areas of surfaces of revolution are determined in this way.

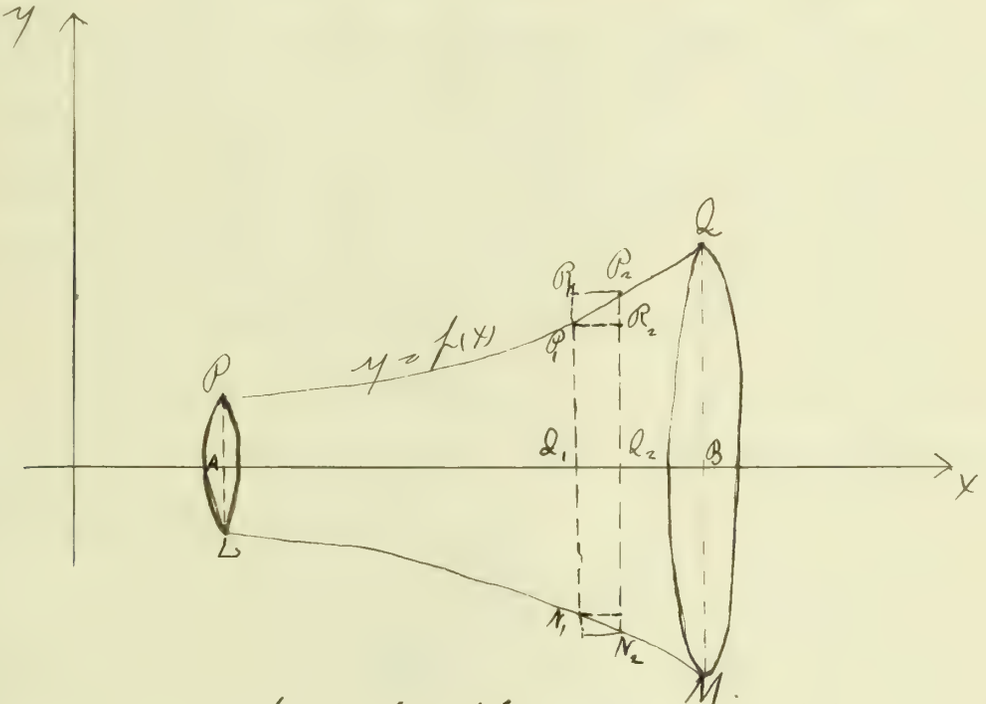
Areas of Curves. It has been shown in the discussion of the definite integral, page 82, that the area between the curve $y = f(x)$, the x-axis and the two ordinates for which $x = x_0$ and $x = x_n$, is expressed by

$$\int_{x_0}^{x_n} f(x)\,dx$$

It has also been shown that this area can

be evaluated by finding the indefinite integral for $f(x)$, substituting x_n and x_0 in turn for x in the indefinite integral, and taking the difference between the results of the two substitutions. It is not necessary to take up the discussion of this particular problem again.

Volumes of solids of revolution. Let PQ be an arc of a curve whose equation is $y = f(x)$, where $f(x)$ is a continuous function. Draw the ordinates AP and BQ and let $OA = x_0$, $OB = x_n$. The volume of the solid



$PQML$, generated by the revolution of $APQB$ about the x-axis is required. In the revolution of PQ each point of the arc PQ

will describe a circle. Suppose AB is divided into n parts, in any way we please. Let $L_1 = X_1$, $L_2 = X_2$. Construct the rectangles $P_1 L_2$, $P_2 L_1$ as indicated in the figure, and suppose that they have revolved about OX with $APLB$.

It is evident that the volume of each plate such as $P_1 R_2 N_2 N_1$ of the solid of revolution is less than the volume of the corresponding exterior cylinder generated by the revolution of the rectangle $L_1 R_1 P_2 L_2$ about the x-axis, and greater than the volume of the corresponding interior cylinder generated by the revolution of the rectangle $L_1 P_1 R_2 L_2$ about the x-axis. Now the volume of the cylinder generated by $L_1 P_1 R_2 L_2$

$$= \pi \overline{P_1 L_1}^2 (L_2 - L_1)$$
$$= \pi \overline{f(X_1)}^2 (X_2 - X_1)$$

and the volume of the cylinder generated by $L_1 R_1 P_2 L_2$

$$= \pi \overline{P_2 L_2}^2 (L_2 - L_1)$$
$$= \pi \overline{f(X_2)}^2 (X_2 - X_1)$$

Since $f(x)$ is a continuous function it has, in the given interval (X_0, X_n) a maximum and a minimum

upon the selection of the positions of the points $x_1, x_2, x_3, \cdots\cdots x_n$. As we shall make this depend upon the definite integral we can assume the same thing here. We need to show that as $n \doteq \infty$ the two summations on the extremes of the above inequality exist and that they are equal. That they each exist, follows from the fact that they vary monotone. It remains now to show that the two summations,

$$\sum_1^n \pi\, G_{n-1}^2 (x_n - x_{n-1}), \quad \sum_1^n \pi\, g_{n-1}^2 (x_n - x_{n-1}),$$

approach the same limit as $n \doteq \infty$, n denoting the number of points inserted between x_0 and x_n.

$$\lim_{n\doteq\infty} \left\{ \sum_1^n \pi\, G_{n-1}^2 (x_n - x_{n-1}) - \sum_1^n \pi\, g_{n-1}^2 (x_n - x_{n-1}) \right\}$$

$$= \lim_{n\doteq\infty} \sum_1^n \pi (G_{n-1}^2 - g_{n-1}^2)(x_n - x_{n-1}).$$

Since $f(x)$ is continuous it is therefore finite and the greatest oscillation $(G_{n-1} - g_{n-1})$, which it can have in any interval is finite. Denote this greatest oscillation by $(\bar{G}_{n-1} - \bar{g}_{n-1})$. We have then

$$\lim_{n\doteq\infty} \sum_1^n \pi (G_{n-1}^2 - g_{n-1}^2)(x_n - x_{n-1}) \leq \lim_{n\doteq\infty} \pi (\bar{G}_{n-1}^2 - \bar{g}_{n-1}^2)(x_n - x_{n-1})$$

$$\leq \lim_{n\doteq\infty} \pi (G_n^2 - g_n^2)(x_n - x_0).$$

value. Let G_1 and g_1 represent the maximum and minimum values in the interval (x_2 , x_1). Now it is evident that the volume of the plate $P_1 P_2 N_2 N_1$ is such that

1.) $\pi G_1^2 (x_2 - x_1) \geq P_1 P_2 N_2 N_1 \geq g_1^2 (x_2 - x_1)$.

Suppose $P_2 M L$ is divided into n plates such as $P_1 P_2 N_2 N_1$, one of which corresponds to each segment of (x_0, x_n), and suppose that the exterior and interior cylinders, corresponding to each of these plates, are constructed. Also let G_2 and g_2 be the largest and smallest values, respectively, of $f(x)$ in the interval $(x_3 - x_2)$ and so in the same way, G_n, and g_{n-1} are the corresponding values of $f(x)$ in $(b - x_{n-1})$. If we take the sum of all the plates $P_1 P_2 N_2 N_1$ we have the volume required. Now taking the sums of the interior and exterior cylinders we have from (1) the following condition. i.e.

2.) $\sum_1^n \pi G_{n-1}^2 (x_n - x_{n-1}) \geq P_2 M L \geq \sum_1^n \pi g_{n-1} (x_n - x_{n-1})$.

We have already shown that the existence of the definite integral does not depend

But as the value of n increases the greatest oscillation $(\bar{G}_{n-1} - \bar{g}_{n-1})$ grows smaller and smaller because the length of the interval grows smaller. Moreover $\bar{G}_{n-1} \doteq \bar{g}_{n-1}$ as $n \doteq \infty$. Hence we have

$$\lim_{n \doteq \infty} \sum_{1}^{n} \pi \bar{G}_{n-1}^2 (x_n - x_{n-1}) = \lim_{n \doteq \infty} \sum_{1}^{n} \pi \bar{g}_{n-1}^2 (x_n - x_{n-1})$$
$$= \int_{x_1}^{x_n} \pi \bar{f}(x)^2 \, dx$$

by definition. Therefore we have

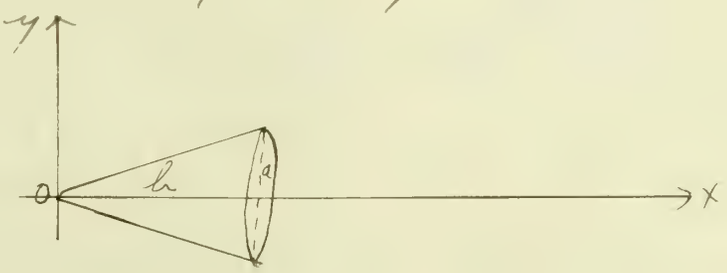
$$\text{Volume of } PdMb = \lim_{n \doteq \infty} \sum_{1}^{n} \pi \bar{G}_{n-1}^2 (x_n - x_{n-1})$$
$$= \lim_{n \doteq \infty} \sum_{1}^{n} \pi \bar{g}_{n-1}^2 (x_n - x_{n-1})$$
$$= \int_{x_1}^{x_n} \pi \bar{f}(x)^2 \, dx.$$

As an illustration it may be well to find such a volume by the general method.

Example. Find the volume of the right cone generated by revolving about the x-axis the line joining the origin and the point (h, a).

Let M be the point (h, a). The equation of OM is

$$ax = hy.$$



Since the value of the definite integral has been shown to be independent of the manner of division of the given interval, we may use the following method. Let

$$q = \sqrt[n]{\frac{h}{\alpha}},$$

where (α, h) is the interval under consideration. Now $\alpha q^n = h$, and the values of the points of division are $\alpha q, \alpha q^2, \alpha q^3, \ldots \alpha q^{n-1}$ and the intervals are $\alpha(q-1), \alpha q(q-1), \alpha q^2(q-1), \ldots \alpha q^{n-1}(q-1)$. Since $ax = by$ we have

$$y = \frac{a}{h} \cdot x$$

and

$$f(x_1) = \frac{a}{h}, \quad f(x_2) = \frac{a}{h} \cdot x_1,$$
$$= \frac{a}{h}\alpha(q-1), \quad = \frac{a}{h} \cdot \alpha q(q-1)$$

etc.

Now we have

$$1) \; S = \pi \frac{a^2}{h^2}\left[\alpha(q-1)\alpha^2 + \alpha q(q-1)(\alpha q)^2 + \alpha q^2(q-1)(\alpha q^2)^2\right.$$
$$\left. + \alpha q^3(q-1)(\alpha q^3)^2 + \ldots + \alpha q^{n-1}(q-1)(\alpha q^{n-1})^2\right]$$
$$= \pi \frac{a^2}{h^2}\alpha^3(q-1)\left[1 + q^3 + q^6 + q^9 + \ldots + q^{2(n-1)+n-1}\right]$$

But $\left[1 + q^3 + q^6 + q^9 + \ldots + q^{2(n-1)+n-1}\right]$ is a geometrical progression whose sum

is

$$S_1 = \frac{q^3 \cdot q^{2(n-1)+n-1} - 1}{q^3 - 1} = \frac{q^{2n+n} - 1}{q^3 - 1} = \frac{q^{3n} - 1}{q^3 - 1}.$$

Since $\alpha q^n = h$, $h^3 = \alpha^3 q^{3n}$, and $q^{3n} = \frac{h^3}{\alpha^3}$. Substituting this value of q^{3n} in S_1 we have

$$S_1 = \frac{\frac{h^3}{\alpha^3} - 1}{q^3 - 1} = \frac{h^3 - \alpha^3}{\alpha^3 (q^3 - 1)}.$$

Now from (1) we have

$$S = \pi \frac{a^2}{h^2} (q-1)(h^3 - \alpha^3)$$

It now remains to prove our division legitimate. We have to show that each of our intervals $\alpha(q-1)$, $\alpha q(q-1)$ ------ $\alpha q^{n-1}(q-1)$ approaches zero as n increases indefinitely. Since

$$q^n = \frac{h}{\alpha}$$

the limiting value of q, as n increases indefinitely, must be 1, otherwise

$$\lim_{n \doteq \infty} q^n$$

would not be finite. Therefore we have

$$\lim_{n \doteq \infty} [\alpha q^K (q-1)] = \lim_{q \doteq 1} [\alpha q^K (q-1)] = 0.$$

Now we have

$$\int_{\alpha}^{h} \pi f(x)^2 dx = \lim_{n \doteq \infty} S = \lim_{q \doteq 1} \left[\pi \frac{a^2}{h^2} \cdot \frac{(q-1)}{(q^3-1)} (h^3 - \alpha^3) \right]$$

$$= \frac{\pi a^2}{h^2} \lim_{q=1} \left[\left(\frac{q-1}{q^3-1} \right) (h^3 - \alpha^3) \right]$$

$$= \frac{\pi a^2}{h^2} \cdot (h^3 - \alpha^3) \lim_{q=1} \left(\frac{q-1}{q^3-1} \right)$$

But

$$\frac{q-1}{q^3-1} = \frac{1}{q^2+q+1}$$

and

$$\lim_{q=1} \frac{1}{q^2+q+1} = \frac{1}{3}$$

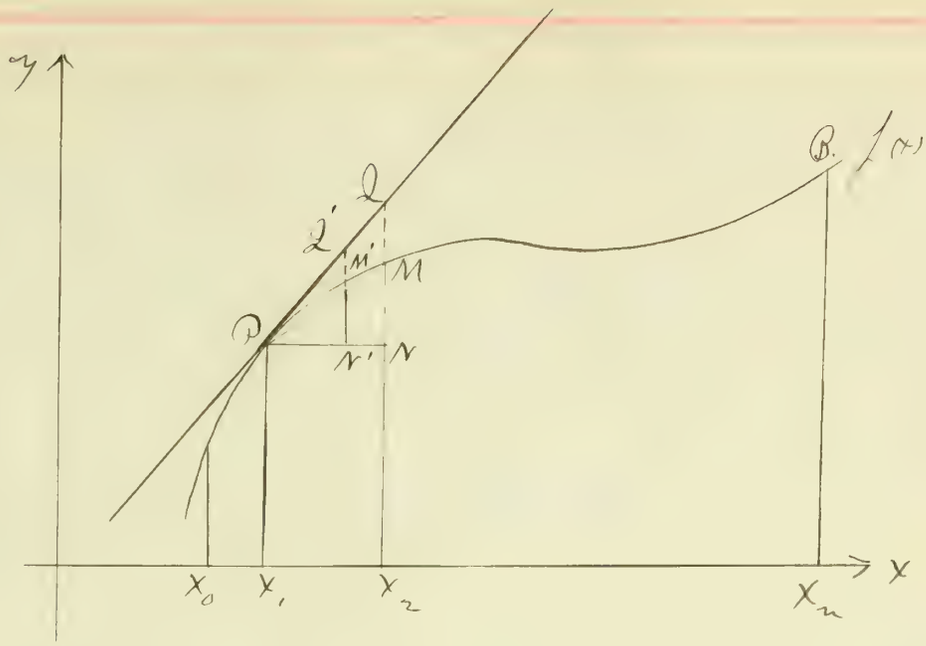
Hence

$$\lim_{n=\infty} S = \frac{\pi a^2}{h^2} (h^3 - \alpha^3) \lim_{q=1} \left(\frac{q-1}{q^3-1} \right) = \frac{\pi a^2}{h^2} \left(\frac{h^3 - \alpha^3}{3} \right)$$

But in this case $\alpha = 0$. Therefore

$$\lim_{n=\infty} S = \frac{\pi a^2 h}{3} = \text{volume of the cone.}$$

Lengths of Curves. To find the length of a curve is equivalent to finding the straight line that has the same length as the curve. The discussion here depends upon the definition that a definite integral is the limit of a sum, the limit being taken for a given interval. Let the equation of the given curve be $f(x, y) = 0$. It is required to find the lengths of the arc AB, A being the point (x_0, y_0) and B, the point (x_n, y_n). On the curve take any two points P and



Mwhose ordinates are x_1 and x_2. Draw the chord PM and the tangent to the curve at the point P i.e. PQ, and make the construction indicated in the figure. We have

1.) $$\overline{PQ}^2 = \overline{PN}^2 + \overline{NQ}^2$$
$$= \overline{PN}^2 + \overline{NM+MQ}^2$$

and

2.) $$\overline{PM}^2 = \overline{PN}^2 + \overline{NM}^2$$

Subtract (2) from (1) and we have

3.) $$\overline{PQ}^2 - \overline{PM}^2 = \overline{NM+MQ}^2 - \overline{NM}^2$$
$$= \left[(NM+MQ) - NM\right]\left[(NM+MQ) + NM\right]$$

Since PQ is drawn tangent to the curve at P its position remains the same no matter how small the sub-divisions of the interval (x_1, x_2) may become. Suppose we divide (x_1, x_2) into n sub-

intervals where n may be made as large as we please. As will be seen from the figure, the distance MQ always decreases in proportion as the value of n increases. Then we may so select n that the value of MQ will approach zero, and we have in the limit

4.) $\lim_{n=\infty} MQ[2NM + MQ] = 0.$

The above expression comes directly from 3) because

$$[(NM + MQ) - NM][(NM + MQ) + NM] = MQ[2NM + MQ].$$

Now the value of the <u>chord</u> PM lies between the values PQ and PM. If we let S_1 represent the arc PM we have the following condition always existing i. e.

5.) $$PQ > S_1 > PM.$$

By (3) and (4) we have

6.) $$\lim_{n=\infty} [\overline{PM}^2 - \overline{PQ}^2] = 0$$

Since the above is true, there is some definite number which is equal to the limiting value of both $\overline{PQ}^2$ and $\overline{PM}^2$. i. e.

$$\lim_{n=\infty} \overline{PM}^2 = \lim_{n=\infty} \overline{PQ}^2 = A.$$

But by (5) this value A is S_1, therefore

7.) $\lim_{n \doteq \infty} PM = \lim_{n \doteq \infty} PQ = S_1$,

By the figure it is easily seen that

8) the chord $PM = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Therefore by (7) we have

9.) $S_1 = \lim_{n \doteq \infty} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

If we wish now to find the length of the curve for the interval (x_0, x_n) we have to sum up the values of the arcs S_1, S_2, S_3 - - - - - S_{n-1}. We may write

10) $S = \sum_{1}^{n-1} S_{n-1}$.

Since $S_{n-1} = \lim_{n \doteq \infty} \sqrt{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2}$, we have

11.) $$S = \lim_{n \doteq \infty} \sum_{1}^{n-1} \sqrt{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2}$$

$$= \lim_{n \doteq \infty} \sum_{1}^{n-1} \sqrt{1 + \left(\frac{y_n - y_{n-1}}{x_n - x_{n-1}}\right)^2} \cdot (x_n - x_{n-1}).$$

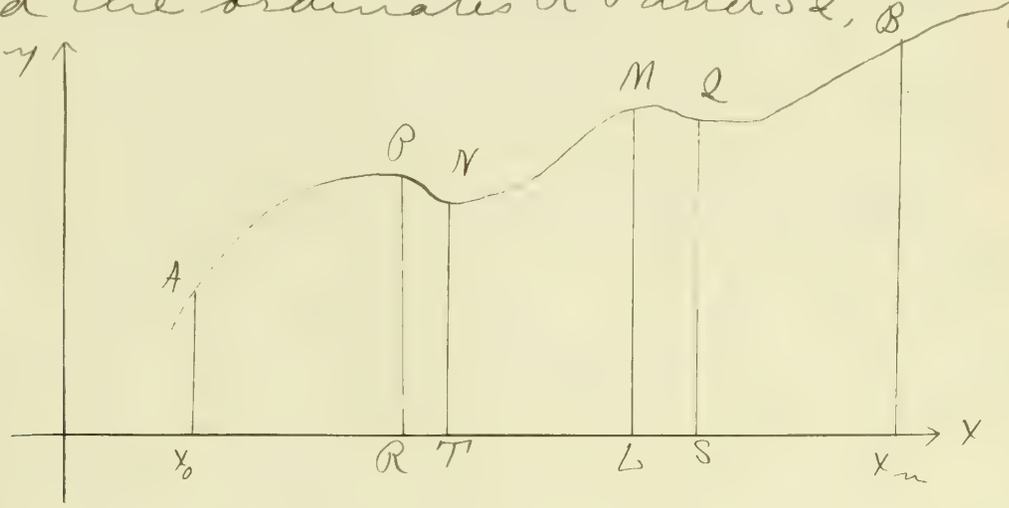
As in definite integrals, we may write (11) as follows. i.e.

$$S = \int_{x_0}^{x_n} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx.$$

Areas of Surfaces of Solids of Revolution.

Suppose that the surface is generated by the revolution, about the x-axis of the arc AB of the curve whose equation is $y = f(x)$; and let the coördinates

of the points A & B be (x_0, y_0) and (x_n, y_n) respectively. Take any two points on the curve, say P and Q, whose coördinates are (x_1, y_1) and (x_2, y_2). Draw the chord PQ and the ordinates RP and SQ, &



and suppose that LM is an ordinate which is not less, and TN is an ordinate which is not greater than any ordinate that can be drawn from the arc PNMQ to the x-axis. On the revolution of the arc AB about OX, each point of AB describes a circle with its ordinate as radius. In this second

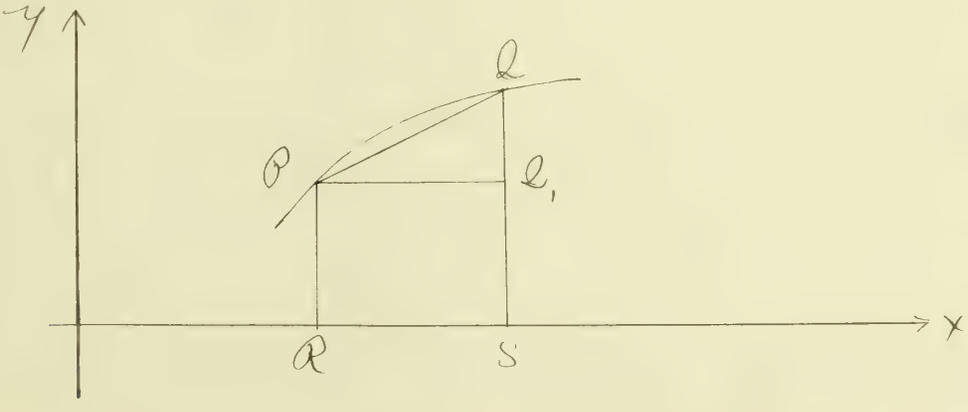


figure suppose L M coincides with S Q and T N coincides with R P. The surface generated by arc P N M Q, we will denote by A_1. We have

1) 2π L M X arc $PQ = \sum_{1}^{n} A_1 = 2\pi$ T N X arc PQ.

We have shown by the preceding article that

$$\underset{n=\infty}{\text{L}} \text{ arc } PQ = \text{chord } PQ,$$

where n is the number of sub-divisions of the interval $(X_2 - X_1)$ and may be made as large as we please. We have also shown that

$$\text{chord } PQ = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x,$$

where $\Delta x = (X_2 - X_1)$ and $\Delta y = (y_2 - y_1)$. Since $f(x)$ is continuous throughout the given interval we have

2) $$\underset{n=\infty}{\text{L}} (G_1 - g_1) = 0,$$

where in this case G_1 = L M, the maximum value of $f(x)$ in the interval $(X_2 - X_1)$, and g_1 = T N, the minimum value. Let G_2 and g_2 be the maximum and minimum values respectively of $f(x)$ in the interval $(X_3 - X_2)$ etc. - - - - G_n and g_n, the corresponding values in the interval $(X_n - X_{n-1})$. Now if we wish to find the area of the surface

generated by the arc A B we must take the sum of those generated by each of the arcs, such as P N M C. Call this S_{n-1}. Then we have

3) $\lim_{n=\infty} \sum_{1}^{n-1} 2\pi g_{n-1} \sqrt{1+\left(\frac{y_n - y_{n-1}}{x_n - x_{n-1}}\right)^2} \cdot (x_n - x_{n-1}) \Big] \doteq S \doteq$

$$\lim_{n=\infty} \sum_{1}^{n-1} \left[2\pi g_{n-1} \sqrt{1+\left(\frac{y_n - y_{n-1}}{x_n - x_{n-1}}\right)^2} \cdot (x_n - x_{n-1}) \right].$$

Then, since (1) is true we have as in the previous discussion.

4) $S = \lim_{n=\infty} \sum_{1}^{n-1} \left[2\pi f(x_{n-1}) \sqrt{1+\left(\frac{y_n - y_{n-1}}{x_n - x_{n-1}}\right)^2} \cdot (x_n - x_{n-1}) \right]$

where $g_{n-1} \doteq f(x_{n-1}) \doteq g_{n-1}$.

We may write (4) as follows

$$S = 2\pi \int_{x_0}^{x_n} f(x) \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot dx.$$

UNIVERSITY OF ILLINOIS-URBANA
3 0112 086762652